UNIT-1

INTRODUCTION

Definition of signal: A signal is a physical quantity that varies with time, space or other independent variables.

A signal can be function of one or more independent variables.

If a signal depends on only one variable, then it is known as one-dimensional signal.

Eg: AC power supply signal and speech signal.

If a signal depends on two independent variables, then the signal is known as two dimensional signal.

Eg: X-ray images

The speed of wind and air pressure are a function of four independent variables: latitude, longitude, elevation and time. These types of signals are known as multi-dimensional signals.

Examples of signals

1. speech signal.
2. ECG signal
3. EEG signal.

→ **Classification of signals**

1. Continous-time, Discrete-time and Digital signals

Continous-time signals: The signals that are defined for every instant of time are known as continous-time signals.

They are denoted by $x(t)$

Discrete-time signals: The signals that are defined at discrete instants of time are known as discrete-time signals. The time are known as discrete-time signals. They are continous in amplitude and discrete in time. They are denoted by $x(n)$.

Digital signals : The signals that are discrete in time and quantized in amplitude are digital signals.

→ Sketch the continuous time signal $x(t) = 2e^{-2t}$ for an interval $0 \le t \le 2$. Sample the continuous-time signal with a sampling period $T = 0.2\,sec$ and sketch the discrete-time signal.

sol). **Given :**

$$x(t) = 2e^{-2t}$$

To sketch the signal we find the values of $x(t)$ for different values of t, ie

$x(0) = 2e^{0} = 2$

$x(0.2) = 2e^{-0.4} = 1.3406$

$x(0.4) = 2e^{-0.8} = 0.8987$

$x(0.6) = 2e^{-1.2} = 0.6024$

$x(0.8) = 2e^{-1.6} = 0.4038$

$x(1) = 2e^{-2} = 0.2707$

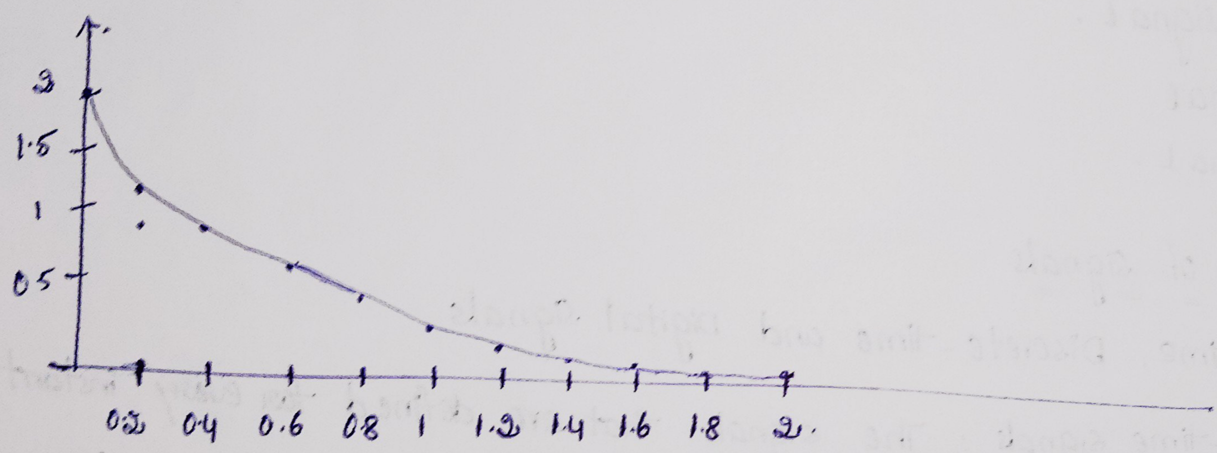
$x(1.2) = 2e^{-2.4} = 0.1814$

$x(1.4) = 2e^{-2.8} = 0.1216$

$x(1.6) = 2e^{-3.2} = 0.0815$

$x(1.8) = 2e^{-3.6} = 0.0546$

$x(2) = 2 = 0.0366$



Given the sampling period $T = 0.2$ sec.

$x(nT) = x(t)|_{t=nT}$

$= x(t)|_{t=0.2n}$

$= x(0.2n)$

$x(n) = 2e^{-2(0.2n)}$

$= 2e^{-0.4n}$

$x(0) = 2$

$x(1) = 2e^{-0.4} = 1.3406$

$x(2) = 2e^{-0.8} = 0.8987$

$x(3) = 2e^{-1.2} = 0.6024$

$x(4) = 2e^{-1.6} = 0.4038.$

$x(5) = 0.2707$

$x(6) = 0.1814$

$x(7) = 0.1216$

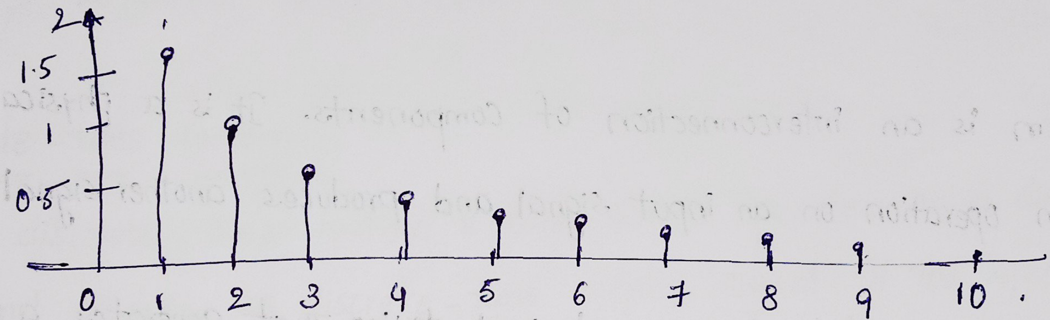
$x(8) = 0.0815$

$x(9) = 0.0546$

$x(10) = 0.0366.$

The sequence $x(n)$ can be written as

$x(n) = \{2, 1.3406, 0.8987, 0.6024, 0.4038, 0.2707, 0.1814, 0.1216, 0.0815, 0.0546, 0.0366\}$.



→ sketch the signal $x(t) = \sin 7t + \sin 10t$ for an interval $0 \le t \le 2$. sample the signal with a sampling period $T = 0.2$sec and sketch the discrete-time signal.

Sol.

→ sketch the signal $x(t) = e^{-t^2/2}$ for $-1 \le t \le 1$. Sample the signal with a sampling period $T = 0.1$sec and sketch the discrete-time signal.

Deterministic and Random signals

A determinstic signal is a signal exhibiting no uncertainity of value at any given instant of time.

Eg : Its instantaneous value can be accurately predicted by mathematical equation. One such signal $x_1(n) = \sin(0.1\pi n)$

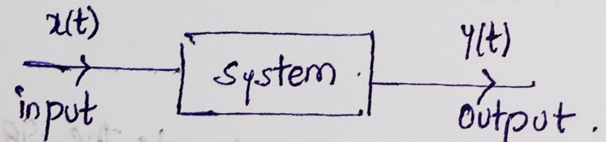
A Random signal is a signal characterized by uncertainity before its actual occurrence.

Eg : Noise.

System

A system is an interconnection of components. It is a physical device that performs an operation on an input signal and produces another signal as output.

Def: A system is defined as a physical device that generates a response or an output signal for a given input signal.



The relationship between the input $x(t)$ and corresponding output $y(t)$ of a system has the form

$$y(t) = \text{operation on } x(t)$$
$$= T[x(t)]$$

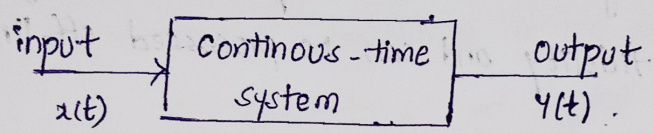
Classification of systems

The systems can be classified as
1. Continous - time system
2. Discrete - time system.

① **Continous - time system**: A Continous - time system is one which operates on a continous - time signal and produces a continous - time output signal.

If the input and output of continous - time systems are $x(t)$, & $y(t)$, then we say that $x(t)$ is tranferred to $y(t)$. That is

$$y(t) = T[x(t)]$$

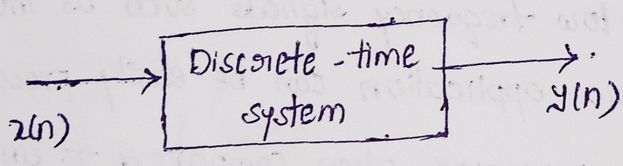


Eg :- Amplifiers, filters, motor etc.

② **Discrete - time system:**

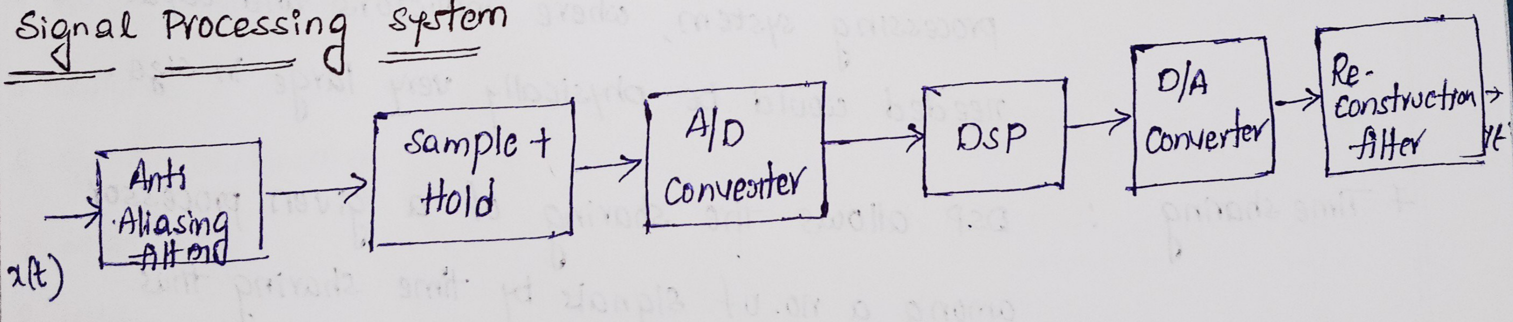
A discrete - time system is one which operates on a discrete - time signal and produces a discrete - time output signal. If the input and output of discrete - time system are $x(n)$ and $y(n)$, then we can write

$$y(n) = T[x(n)]$$



Digital

→ **Signal Processing system**



→ Advantages of Digital Signal Processor

1. Greater accuracy : The tolerance of the circuit components used to design the analog filters affects the accuracy, whereas the DSP provides superior control of accuracy.

2. Cheaper : In many applications, digital realization is comparatively cheaper than its analog counterpart.

3. Ease of Data storage : Digital signals can be easily stored on magnetic media without loss of fidelity and can be processed off-line in a remote laboratory.

4. Implementation of sophisticated algorithms : The DSP allows to implement sophisticated algorithms when compared to its analog counterpart.

5. Flexibility in Configuration : A DSP system can be easily reconfigured by changing the program. Reconfiguration of an analog system involves redesign of system Hardware.

6. Applicability of VLF signals : The very low frequency signals such as those occuring in seismic application can be easily processed using a digital processor when compared to an analog processing system, where inductors and capacitors needed would be physically very large in size.

7. Time sharing : DSP allows the sharing of a given processor among a no. of signals by time sharing thus reducing the cost of processing a signal.

Limitations

1. System Complexity : System complexity increases in the digital processing of an analog signal because of devices such as A/D & D/A converters and their associated filters.

2. Bandwidth Limited by sampling rate : Band Limited signals can be sampled without information loss if the sampling rate is more than twice the bandwidth. Therefore, signals having extremely wide bandwidth require fast sampling rate A/D converters and fast digital signal processors. But there is a practical limitation in the speed of operation of A/D converters and digital signal processors.

3. Power Consumption : A variety of analog processing algorithms can be implemented using passive circuit elements like inductors, capacitors and resistors that donot need much power, whereas the DSP chip containing over 4 lakh transistors dissipates more power (1 Watt).

Applications of DSP

1. Telecommunication
2. Consumer Electronics
3. Instrumentation and control
4. Image processing
5. Medicine
6. Speech Processing
7. Seismology
8. Military

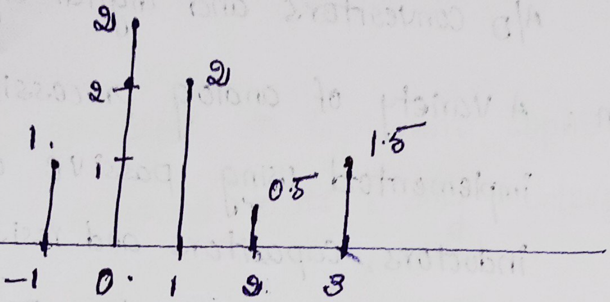
Representation of Discrete-time signals:

There are different types of representations for discrete-time signals. They are

1. Graphical Representation
2. Functional Representation
3. Tabular Representation
4. Sequence Representation.

Graphical Representation:

Let us consider a signal $x(n)$ with values $x(-1) = 1$; $x(0) = 2$; $x(1) = 2$; $x(2) = 0.5$ and $x(3) = 1.5$. This discrete-time signal can be represented graphically as shown in Fig:



Functional Representation:

The discrete-time signal can be represented using functional representation as below

$$x(n) = \begin{cases} 1 & \text{for } n = -1 \\ 2 & \text{for } n = 0 \\ 0.5 & \text{for } n = 2 \\ 1.5 & \text{for } n = 3 \\ 0 & \text{otherwise} \end{cases}$$

Tabular Representation

The discrete-time signal can also be represented as

n	-1	0	1	2	3
$x(n)$	1	2	2	0.5	1.5

Sequence representation.

A finite duration sequence with time origin. $(n = 0)$ indicated by the symbol ↑ is represented as

$$x(n) = \{1, \underset{\uparrow}{2}, 2, 0.5, 1.5\}$$

An infinite duration sequence can be represented as

$$x(n) = \{\ldots,\ 0.2,\ 1,\ -1,\ 3,\ 2,\ \ldots\}$$

A finite duration sequence that satisfies the condition $x(n) = 0$ for $n < 0$ can be represented as

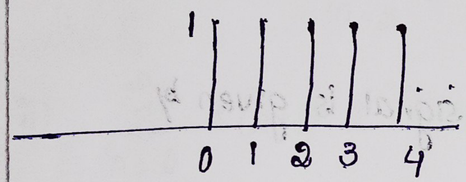
$$x(n) = \{2, 4, 6, 8, -3\}$$

→ **Elementary Discrete-time signals.**

1. **Unit step sequence:**

It is defined as

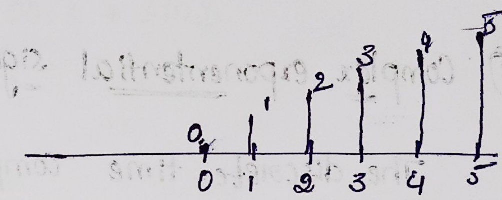
$$u(n) = 1 \text{ for } n \geq 0$$
$$= 0 \text{ for } n < 0.$$



2. **Unit ramp sequence**

It is defined as

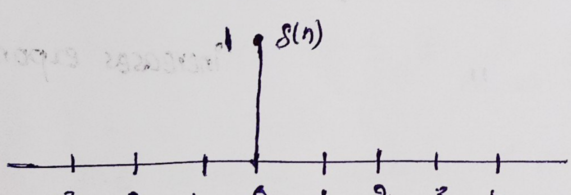
$$r(n) = n \text{ for } n \geq 0$$
$$= 0 \text{ for } n < 0$$



3. **Unit sample sequence (unit impulse response).**

The unit sample sequence is defined as

$$\delta(n) = 1 \text{ for } n = 0$$
$$= 0 \text{ for } n \neq 0$$



4. **Exponential sequence**

It is defined as

$$x(n) = a^n \text{ for all } n$$

When the value $a > 1$, the sequence grows exponentially.

When the value is $0 < a < 1$ the sequence decays exponentially.

Properties.

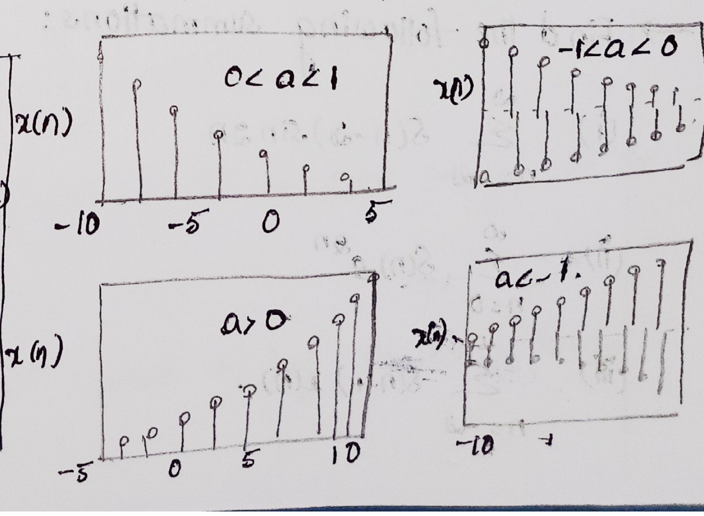
① $\delta(n) = u(n) - u(n-1)$

$$u(n) = \sum_{k=-\infty}^{\infty} \delta(k)$$

② $\sum_{k=-\infty}^{\infty} x(n)\,\delta(n-n_0) = x(n_0)$

③ $\delta(n-k) = \begin{cases} 1 & \text{for } n = k \\ 0 & \text{for } n \neq k. \end{cases}$

④ $x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)$



(5) Sinusoidal Signal:

A discrete-time sinusoidal signal is given by

$$x(n) = A\cos(\omega_0 n + \phi)$$

where ω_0 is the frequency in radians per sample

ϕ is the phase in radians

Using Euler's identity, we can write

$$A\cos(\omega_0 n + \phi) = \frac{A}{2} e^{j\phi} e^{j\omega_0 n} + \frac{A}{2} e^{-j\phi} e^{-j\omega_0 n}$$

since $|e^{j\omega_0 n}|^2 = 1$, the energy of the signal is infinite

Avg power of the signal is 1

(6) Complex exponential signal:

The discrete-time complex exponential signal is given by

$$x(n) = a^n e^{j(\omega_0 n + \phi)}$$

$$= a^n \cos(\omega_0 n + \phi) + j a^n \sin(\omega_0 n + \phi)$$

For $|a| = 1$, the real and imaginary parts of complex exponential sequences are sinusoidal

$|a| < 1$, the amplitude of the sinusoidal sequence decays exponentially

$|a| > 1$, the amplitude of the " increases exponentially.

→ Find the following summations:

(i) $\sum_{n=-\infty}^{\infty} \delta(n-2)\sin 2n$

(ii) $\sum_{n=0}^{\infty} \delta(n) e^{2n}$

(iii) $\sum_{n=-\infty}^{\infty} \delta(n+1)\, x(n)$.

i) $\sum_{n=-\infty}^{\infty} \delta(n-2)\sin 2n = \sin 2n \big|_{n=2}$

$= \sin 4.$

ii) $\sum_{n=0}^{\infty} \delta(n) e^{2n} = e^{2n}\big|_{n=0} = 1$

iii) $\sum_{n=-\infty}^{\infty} \delta(n+1) x(n) = x(n)\big|_{n=-1} = x(-1).$

→ Find the following summations:

(i) $\sum_{n=-\infty}^{5} [\delta(n-2)\cos 2n + \delta(n-1)\sin 2n]$

$= \sum_{n=-\infty}^{5} \delta(n-2)\cos 2n + \sum_{n=-\infty}^{5} \delta(n-1)\sin 2n$

$= \cos 2n\big|_{n=2} + \sin 2n\big|_{n=1} = \cos 4 + \sin 2.$

ii) $\sum_{n=0}^{\infty} \delta(n+1) e^{-2n} = e^{-2n}\big|_{n=-1}$

$= e^{2} = 7.3$

→ Classification of Discrete-Time signals:

1. Energy signals and Power signals:

For a discrete-time signal $x(n)$ the Energy E is defined as

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

The average power of a discrete-time signal $x(n)$ is defined as

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

A signal is an energy signal, if and only if the total energy of the signal is finite. For an energy signal $P=0$.

Illy the signal is said to be power signal if Avg Power is finite. and for power signal $E=\infty$.

→ Determine the values of power and Energy of the following. Find whether the signals are power, energy or neither energy nor Power signals.

Sol: (i) $x(n) = \left[\frac{1}{3}\right]^n u(n)$

The Energy of the signal.

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$u(n) = 1$ for $n \geq 0$
$= 0$ for $n < 0$.

$$= \sum_{n=0}^{\infty} \left[\left(\frac{1}{3}\right)^n\right]^2 = \sum_{n=0}^{\infty} \left(\frac{1}{9}\right)^n$$

$1 + a + a^2 + \dots + \infty = \frac{1}{1-a}$

$$= \frac{1}{1-\frac{1}{9}} = \frac{1}{\frac{9-1}{9}} = \frac{9}{8}$$

$$P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=0}^{N} \left(\frac{1}{9}\right)^n$$

$\sum_{k=0}^{n} ar^k = a\left(\frac{1-r^n}{1-r}\right).$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \left(\frac{1-\left(\frac{1}{9}\right)^n}{1-\frac{1}{9}}\right)$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \left(\frac{1-\left(\frac{1}{9}\right)^n}{\frac{8}{9}}\right)$$

$= 0$

The energy is finite and Power is zero. Therefore, the signal is an Energy signal.

Note: The Geometric Sum Formula:

Sum of finite terms $= \frac{a(1-r^n)}{1-r}$; $a, ar, ar^2, ar^3 \dots \infty$

Sum of infinite terms $= \frac{a}{1-r}$; $a, ar, ar^2, ar^3 \dots ar^{N-1}$

ii) $x(n) = e^{j\left(\frac{\pi}{2}n + \frac{\pi}{4}\right)}$.

$$E = \sum_{n=-\infty}^{\infty} \left| e^{j\left(\frac{\pi}{2}n + \frac{\pi}{4}\right)} \right|^2 \quad \because e^{j(\omega+\theta)} = 1$$

$$= \sum_{n=-\infty}^{\infty} 1 = \infty$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \left| e^{j\left(\frac{\pi}{2}n + \frac{\pi}{4}\right)} \right|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} 1$$

$$= \lim_{N\to\infty} \frac{1}{\cancel{2N+1}} (\cancel{2N+1}) \quad \left[\because \sum_{n=-N}^{N} 1 = 2N+1\right).$$

$$= 1$$

Therefore Energy is infinite and Power is finite. Therefore the signal is a power signal.

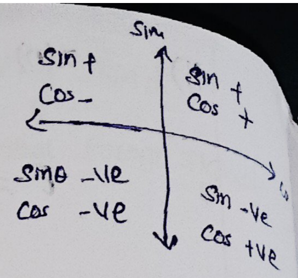
iii) $x(n) = \sin\left(\frac{\pi}{4}n\right)$

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=-\infty}^{\infty} \left| \sin\left(\frac{\pi}{4}n\right)\right|^2$$

$$= \sum_{n=-\infty}^{\infty} \left| \frac{1 - \cos 2 \cdot \frac{\pi}{4}n}{2} \right| = \sum_{n=-\infty}^{\infty} \left[\frac{1 - \cos\frac{\pi}{2}n}{2}\right] = \infty.$$

=



$$P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} \left| \sin^2\left(\frac{\pi}{4}n\right) \right|$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} \left(\frac{1-\cos\frac{\pi}{2}n}{2} \right)$$

$$\frac{1}{2} \underset{N\to\infty}{Lt} \frac{1}{2N+1} \left[\sum_{n=-N}^{N} 1 \right.$$

$$= \frac{1}{2} \underset{N\to\infty}{Lt} \frac{1}{2N+1} \cdot 2N+1 = \frac{1}{2}$$

The Energy is infinite and power is finite. Therefore the signal is a power signal.

(iv) $x(n) = e^{2n} u(n)$

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=0}^{\infty} |e^{2n}|^2 = \sum_{n=0}^{\infty} e^{4n} = 1 + e^4 + e^8 + \cdots + \infty = \infty$$

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=0}^{N} |e^{2n}|^2$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=0}^{N} e^{4n} =$$

$$= \infty$$

The signal is neither Energy nor Power signal.

→ Find whether the signals are power, energy or neither Power nor Energy signals.

(i) $\cos(\omega_0 n)\, u(n)$.

As $t \to \infty$ Mag $\to 0$ (Energy)

As $t \to \infty$ Mag $\neq 0$ (Power)

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=0}^{\infty} \cos^2 \omega_0 n = \sum_{n=0}^{\infty} \left(\frac{1+\cos 2\omega_0 n}{2}\right)$$

(All Power signals are need not to be a periodic)

$$= \frac{1}{2}\sum_{n=0}^{\infty} 1 + \frac{1}{2}\sum_{n=0}^{\infty} \cos 2\omega_0 n$$

(but all periodic signals are power signals)

$$= \frac{1}{2}(\infty) \qquad = \infty.$$

$$P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

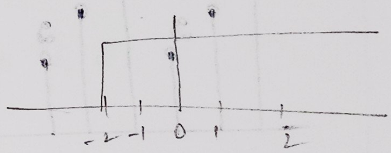
$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=0}^{N} \cos^2 \omega_0 n = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=0}^{N} \left(\frac{1+\cos 2\omega_0 n}{2}\right)$$

$$= \frac{1}{2} \underset{N\to\infty}{Lt} \frac{1}{2N+1} \cdot 2N+1 = \frac{1}{2}$$

$n+2=0$

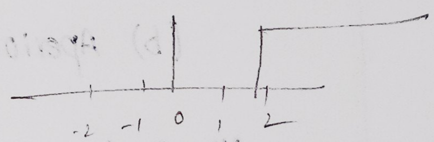
$n=-2$

(ii) $u(n+2) - u(n-2)$



$x(n) = 1$ for $-2 \le n \le 2$

$= 0$ for otherwise



$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=-2}^{2} 1 = 5$$

Energy is finite

Power is zero then

it is called Energy signal.

$$P = \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

$$= \underset{N\to\infty}{Lt} \frac{1}{2N+1} \sum_{n=-2}^{2} 1 = 0$$

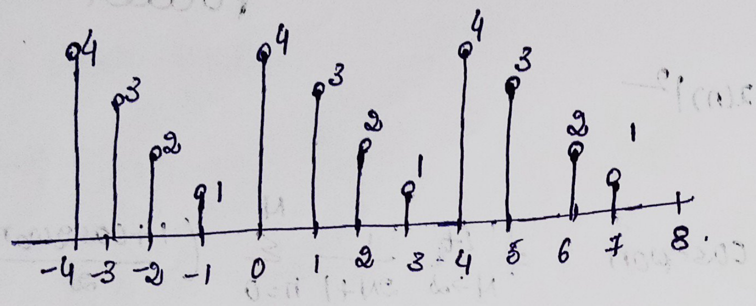
Periodic and Aperiodic signals :

A discrete-time signal $x(n)$ is said to be periodic with period 'N' if and only

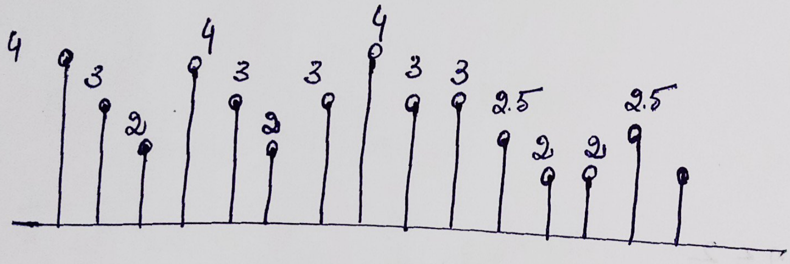
$$x(N+n) = x(n) \text{ for all } n \quad -①$$

The smallest value of N for which eq ① holds is known as Fundamental Period.

If eq ① does not satisfy even for one value of n, then the discrete time signal is Aperiodic.



(a) Periodic signal



(b) Aperiodic signal

All continuous-time sinusoidal signals with fundamental frequency Ω are periodic. In contrast, all the discrete-time sinusoidal sequences are not periodic. Consider a discrete-time signal given by

$$x(n) = A \sin(\omega_0 n + \theta) \quad -②$$

where A is Amplitude

ω_0 and θ are frequency and phase shift respectively.

A discrete-time sequence is periodic if it satisfies the condition

$$x(n+N) = x(n) \quad — ③$$

$$= A\sin(\omega_0(n+N)+\theta)$$

$$= A\sin(\omega_0 n + \omega_0 N + \theta)$$

For $x(n)$, eq ③ satisfies if and only if $\omega_0 N$ is an integer multiple of 2π i.e

$$\omega_0 N = 2\pi m$$

$$\omega_0 = 2\pi\left(\frac{m}{N}\right) \quad (\text{or}) \quad N = 2\pi\left(\frac{m}{\omega_0}\right) \quad — ④$$

From eq ④, we find that for the discrete-time signal to be periodic, the fundamental frequency ω_0 must be a rational multiple of 2π. otherwise the discrete-time signal is aperiodic.

→ Determine whether or not each of the following signals is periodic. If a signal is periodic, specify its fundamental period.

(i) $x(n) = e^{j6\pi n}$

$x(n) = e^{j6\pi n}$ $\quad \omega_0 = 6\pi$, The fundamental frequency is multiple of π.

Therefore, the signal is periodic.

$$N = 2\pi\left(\frac{m}{\omega_0}\right)$$

The minimum value of 'm' for which 'N' is integer is 3

$$N = 2\pi \cdot \left(\frac{3}{6\pi}\right) = 1$$

Therefore, the fundamental period = 1

(ii) $x(n) = e^{j\frac{3}{5}\left(n+\frac{1}{2}\right)}$

$$x(n) = e^{j\frac{3}{5}\left(n+\frac{1}{2}\right)}$$

Here $\omega_0 = \frac{3}{5}$ which is not a multiple of π. Therefore, the signal is Aperiodic.

~~The fundamental period~~ $\cancel{N = 2\pi\left(\frac{m}{\omega_0}\right)}$

$$\cancel{= 2\pi\left(\frac{2}{3/5}\right)}$$

(iii) $x(n) = \cos\left(\frac{2\pi}{3}\right)n$

$x(n) = \cos\left(\frac{2\pi}{3}\right)n$, $\omega_0 = \frac{2\pi}{3}$ The signal is periodic

The fundamental period is $N = 2\pi\left(\frac{m}{\omega_0}\right)$

$$= 2\pi\left(\frac{m}{\frac{2\pi}{3}}\right) = 3m$$

for $m = 1$ then $N = 3$

Therefore, the fundamental period of the signal is 3

(iv) $x(n) = \cos\frac{\pi}{3}n + \cos\frac{3\pi}{4}n$

The fundamental period of the signal $\cos\left(\frac{\pi}{3}n\right)$ is

$$N_1 = 2\pi\left(\frac{m}{\pi/3}\right) = 6m$$

for $m = 1 \Rightarrow N_1 = 6$

The minimum integer value of m for which N_1 is integer is 1

Similarly,

$$N_2 = 2\pi\left(\frac{m}{\frac{3\pi}{4}}\right)$$

$$= 8 \times \frac{m}{3} = 8 \quad \text{for } m = 3$$

$$2 \,\underline{|\, 6, 8}$$
$$3, 4$$

$$\therefore \frac{N_1}{N_2} = \frac{6}{8} = \frac{3}{4}$$

$N = 4N_1 = 3N_2 = 24.$

$N = 24.$

→ Determine the fundamental period of the following signals, if they are periodic

(i) $x(n) = \sin\left(\frac{\pi n}{4}\right)$

$\omega_0 = \frac{\pi}{4}$

$N = 2\pi\left(\frac{m}{\omega_0}\right) = 2\pi\left(\frac{m}{\frac{\pi}{4}}\right) = 8m$

$N = 8$ for $m = 1$.

The fundamental period is 8

(ii) $x(n) = e^{j2n}$

$\omega_0 = 2$ which is not multiple of π. Therefore the signal is Aperiodic.

(iii) $x(n) = \cos\frac{\pi}{4}n + \cos 2n$

$N_1 \Rightarrow \omega_1 = \frac{\pi}{4}$

$N_1 = 2\pi\left(\frac{m}{\omega_1}\right) = 2\pi\left(\frac{m}{\frac{\pi}{4}}\right) = 8m$

$N_1 = 8$

$N_2 \Rightarrow \omega_2 = 2$ which is not an integral multiple of π. so the signal is Aperiodic.

→ Symmetric (even) and Antisymmetric (odd) signals

A discrete-time signal $x(n)$ is said to be a symmetric (even) signal if it satisfies the condition

$$x(-n) = x(n) \text{ for all } n$$

Example : $x(n) = \cos\omega n$

The signal is said to be an odd signal if it satisfies the condition

$$x(-n) = -x(n) \text{ for all } n$$

Example : $A\sin\omega n$; If $x(n)$ is odd then $x(0) = 0$

A signal $x(n)$ can be expressed as the sum of even and odd components. That is

$$x(n) = x_e(n) + x_o(n) \quad \text{--- (1)}$$

where $x_e(n)$ is even component of the signal

$x_o(n)$ is odd component of the signal

Replace 'n' by '-n' then eq (1) can be written as

$$x(-n) = x_e(-n) + x_o(-n)$$

$$x(-n) = x_e(n) - x_o(n) \quad \text{--- (2)}$$

Add (1) & (2) then we get

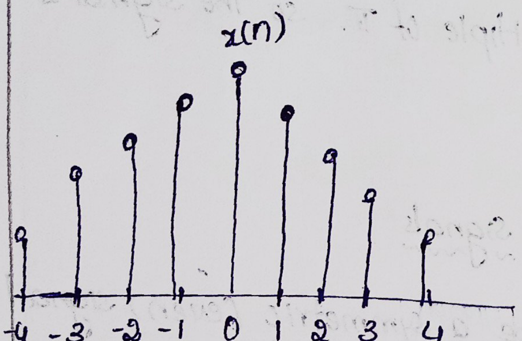
$$x(n) + x(-n) = 2x_e(n)$$

$$x_e(n) = \frac{1}{2}[x(n) + x(-n)]$$

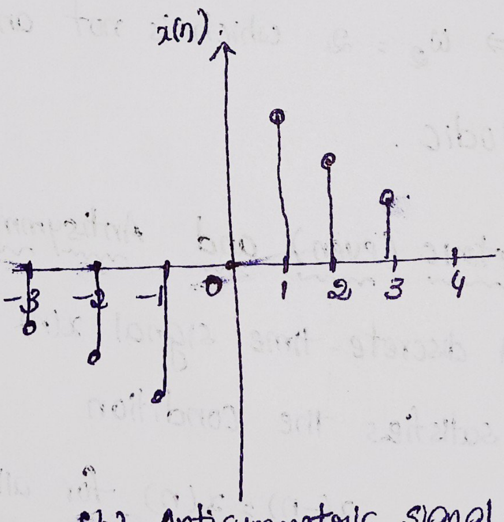
Similarily subtract (1) & (2) then we get

$$x(n) - x(-n) = 2x_o(n)$$

$$x_o(n) = \frac{1}{2}[x(n) - x(-n)]$$



(a) symmetric signal



(b) Antisymmetric signal.

→ Causal and Non-Causal signals

A signal $x(n)$ is said to be causal if its value is zero for $n<0$. otherwise the signal is Non-causal

Examples for causal signals :

$$x_1(n) = a^n u(n)$$

$$x_2(n) = \{1, 2, -3, -1, 2\}$$

Examples for Non-causal signals :

$$x_1(n) = a^n u(-n+1)$$

$$x_2(n) = \{1, -2, \underset{\uparrow}{1}, 4, 3\}$$

A signal that is zero for all $n \geq 0$ is called an anticausal signal.

→ Operation on signals

Signal processing is a group of basic operations applied to an input signal resulting in another signal as the output. The mathematical transformation from one signal to another is represented as

$$y(n) = T[x(n)]$$

The basic set of operations are

1. shifting
2. Time Reversal
3. Time scaling
4. Scalar Multiplication
5. Signal Multiplier
6. Signal addition

→ **shifting**

The shift operation takes the input sequence and shifts the values by an integer increment of the independent variable. The shifting may delay or advance the sequences in time. Mathematically this can be represented as

$$y(n) = x(n-k)$$

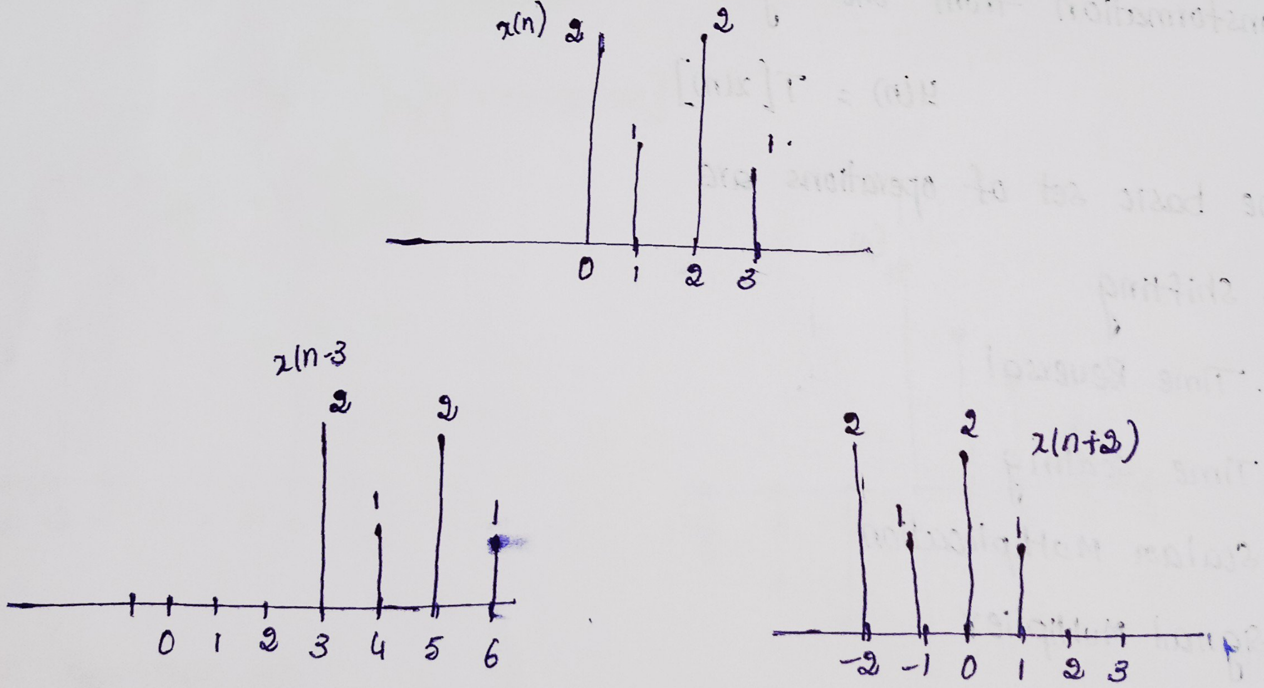
where $x(n)$ is the input

$y(n)$ is the output

If 'k' is positive, the shifting delays the sequence

If 'k' is negative, the shifting advances the sequence.

A signal $x(n)$ as shown in figure. The signal $x(n-3)$ is obtained by shifting $x(n)$ right by 3 units of time. The result is shown in Fig (b). On the other hand, the signal $x(n+2)$ is obtained by shifting $x(n)$ left by two units of time as shown in fig 1(c)



→ **Time scaling**

This is accomplished by replacing n by λn in the sequence $x(n)$.

Let $x(n)$ is a sequence shown in Fig. If $\lambda = 2$ we get a new sequence

$$y(n) = x(2n)$$

We can plot the sequence $y(n)$ by substituting different values of 'n'

For

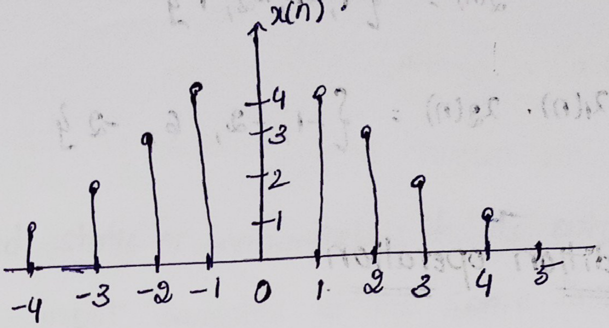
$n = -1$; $y(-1) = x(-2) = 3$

Similarly

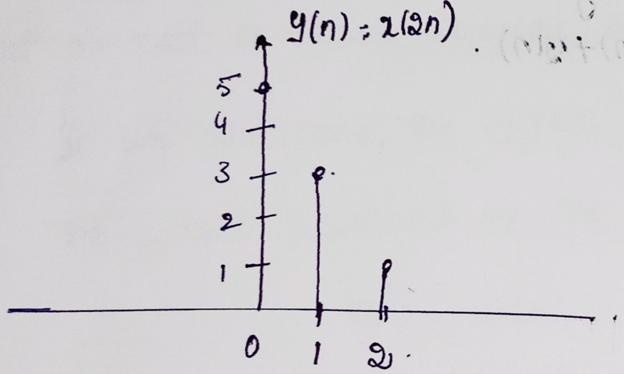
$y(0) = x(0) = 5$

$y(1) = x(2) = 3$

$y(2) = x(4) = 1$.



$y(3) =$



→ **Scalar Multiplication**

A Scalar Multiplier is shown in Fig. Here the signal $x(n)$ is multiplied by a Scale factor 'a'

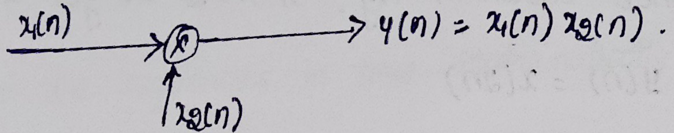
$$x(n) \xrightarrow{a} y(n) = ax(n)$$

For example if $x(n) = \{1, 2, 1, -1\}$ and $a = 2$. Then the signal

$ax(n) = \{2, 4, 2, -2\}$

Signal Multiplier

The multiplication of two signal sequences to form another sequence



For example, if

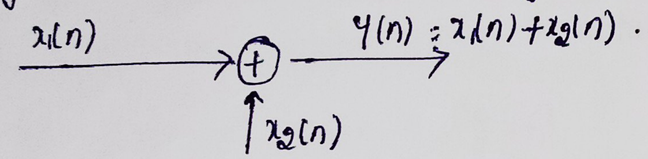
$x_1(n) = \{-1, 2, -3, -2\}$

$x_2(n) = \{1, -1, -2, 1\}$

$x_1(n) \cdot x_2(n) = \{-1, -2, 6, -2\}$

Addition operation

Two signals can be added by using an adder shown in fig.



For example, if

$x_1(n) = \{1, 2, 3, 4\}$ and

$x_2(n) = \{4, 3, 2, 1\}$.

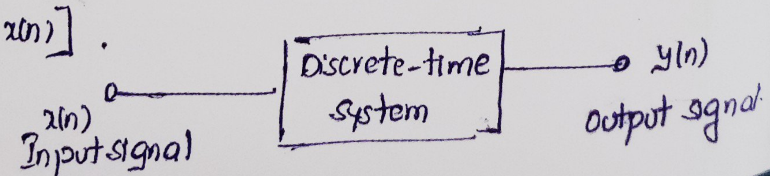
$x_1(n) + x_2(n) = \{5, 5, 5, 5\}$

Discrete-time system

A discrete-time system is a device or an algorithm that operates on a discrete-time input signal $x(n)$, according to some well defined rule, to produce another discrete time signal $y(n)$ called the output signal.

The relationship between $x(n)$ and $y(n)$ is

$$y(n) = T[x(n)].$$



Classification of Discrete-time systems

Discrete-time systems are classified according to their general properties and characteristics. They are

1. Static and Dynamic systems
2. Causal and Non-Causal systems
3. Linear and Non-Linear system
4. Time Varient and Time-Invarient systems
5. FIR and IIR systems
6. Stable and Unstable systems.

Static and Dynamic systems

A discrete-time system is called static or memoryless if its output at any instant 'n' depends on the input samples at the same time, but not on past or future samples of the input.

In any other case, the system is said to be dynamic or to have memory.

The systems described by the following equations are static

$$y(n) = ax(n)$$

$$y(n) = ax^2(n)$$

On the other hand, the systems described by the following equations

$$y(n) = x(n-1) + x(n-2)$$

$$y(n) = x(n) + x(n+1)$$ are dynamic systems.

→ Find whether the following systems are static or dynamic

(i) $y(n) = x(n)\,x(n-1)$

The output $y(n)$ depends on the past input. Therefore the system is Dynamic.

ii) $y(n) = x^2(n) + x(n)$

The output $y(n)$ depends on the input at that instant only. Therefore the system is static.

iii) $y(n) = x(2n)$

The output $y(n)$ depends on the ~~future~~ past input. Therefore the system is Dynamic.

iv) $y(n) = x^2(n)$

The output $y(n)$ depends on the present input i.e at that instant only. Therefore the system is static.

→ Causal and Non-Causal systems:

A system is said to be causal if the output of the system at any time 'n' depends only at present and past inputs, but does not depend on the future inputs. This can be represented mathematically as

$$y(n) = F[x(n), x(n-1), x(n-2) \ldots]$$

If the output of a system depends on future inputs, then system is said to be Non-Causal or Anticipatory.

Example: $y(n) = x(n) + x(n-1)$ Causal system

$y(n) = x(2n)$ Non-Causal system

→ Determine if the system described by the following equations are Causal or Non-Causal

(i) $y(n) = x(n) + \dfrac{1}{x(n-1)}$

The output depends on present and past inputs. Therefore the system is Causal.

(ii) $y(n) = x(n^2)$

For $n = -1$; $y(-1) = x(1)$

$n = 0$; $y(0) = x(0)$

$n = 1$; $y(1) = x(1)$

For all values of 'n' except for n=0 and n=1 the system depend on future inputs. so the system is Non-Causal

(iii) $y(n) = Ax(n) + B$

The output depends on the present input. so the system is Causal.

(iv) $y(n) = ax(n) + bx(n-1)$

The output depends on present and past inputs. so the system is called Causal.

→ **Linear and Non-Linear systems**

A system that satisfies the superposition principle is said to be a Linear system.

Superposition principle states that the response of the system to a weighted sum of signals should be equal to the corresponding weighted sum of the outputs of the system to each of the individual input signals.

A System is linear if and only if

$T[a_1x_1(n) + a_2x_2(n)] = a_1T[x_1(n)] + a_2T[x_2(n)]$ for any arbitrary constants a_1 and a_2.

A system that does not satisfy the superposition principle is called Non-Linear system.

→ Determine if the system described by the following input-output equations is linear or Non-Linear.

(i) $y(n) = x(n) + \frac{1}{x(n-1)}$

Given $y(n) = x(n) + \frac{1}{x(n-1)}$

For two input sequences $x_1(n)$ and $x_2(n)$ the corresponding outputs are

$$y_1(n) = T[x_1(n)] = x_1(n) + \frac{1}{x_1(n-1)} \quad \text{— (1)}$$

$$y_2(n) = T[x_2(n)] = x_2(n) + \frac{1}{x_2(n-1)} \quad \text{— (2)}$$

The output due to weighted sum of inputs is

$$y_3(n) = T[a_1x_1(n) + a_2x_2(n)] =$$

$$= a_1x_1(n) + a_2x_2(n) + \frac{1}{a_1x_1(n-1) + a_2x_2(n-1)} \quad \text{— (3)}$$

On the other hand, the linear combination of the two outputs is.

$$a_1y_1(n) + a_2y_2(n) = a_1x_1(n) + \frac{a_1}{x_1(n-1)} + a_2x_2(n) + \frac{1}{a_2x_2(n-1)} \quad \text{— (4)}$$

The equations (3) & (4) are not equal and the superposition principle is not satisfied. So the system is Non-Linear

(ii) $y(n) = x^2(n)$

The outputs due to the signals $x_1(n)$ and $x_2(n)$ are

$$y_1(n) = T[x_1(n)] = x_1^2(n)$$

$$y_2(n) = T[x_2(n)] = x_2^2(n)$$

The weighted sum of outputs is

$$a_1 T[x_1(n)] + a_2 T[x_2(n)] = a_1 x_1^2(n) + a_2 x_2^2(n) \quad \text{— ①}$$

The output due to weighted sum of inputs is

$$y_3(n) = T[a_1 x_1(n) + a_2 x_2(n)] = [a_1 x_1(n) + a_2 x_2(n)]^2 \quad \text{— ⑤}$$

The eq ① & ⑤ are not equal and the superposition principle is not satisfied. So the system is <u>Non-Linear</u>

(iii) $y(n) = n x(n)$

We have $y_1(n) = T[x_1(n)] = n x_1(n) \quad \text{— ①}$

$y_2(n) = T[x_2(n)] = n x_2(n) \quad \text{— ②}$

The weighted sum of outputs is

$$a_1 T[x_1(n)] + a_2 T[x_2(n)] = a_1 n x_1(n) + a_2 n x_2(n) \quad \text{— ③}$$

The output due to weighted sum of inputs is

$$y_3(n) = T[a_1 x_1(n) + a_2 x_2(n)]$$
$$= a_1 n x_1(n) + a_2 n x_2(n) \quad \text{— ④}$$

The equations ③ & ④ are equal. Superposition principle is satisfied.

Hence, the system is <u>Linear</u>.

(iv) $y(n) = 2x(n) + \dfrac{1}{x(n-1)}$ Ans: Non-Linear.

(v) $y(n) = n x^2(n)$ Ans: Non-Linear

Time Varient and Time - Invarient systems

A system is said to be time - invarient or shift invarient if the characteristics of the system do not change with time.

For a time-invarient system if $y(n)$ is the response of the system to the input $x(n)$, then the response of a system to the input $x(n-k)$ is $y(n-k)$.

In otherwords, if the input sequence is shifted by 'k' samples, the generated output sequence is the original sequence shifted by 'k' samples. This concept is illustrated in below figure.

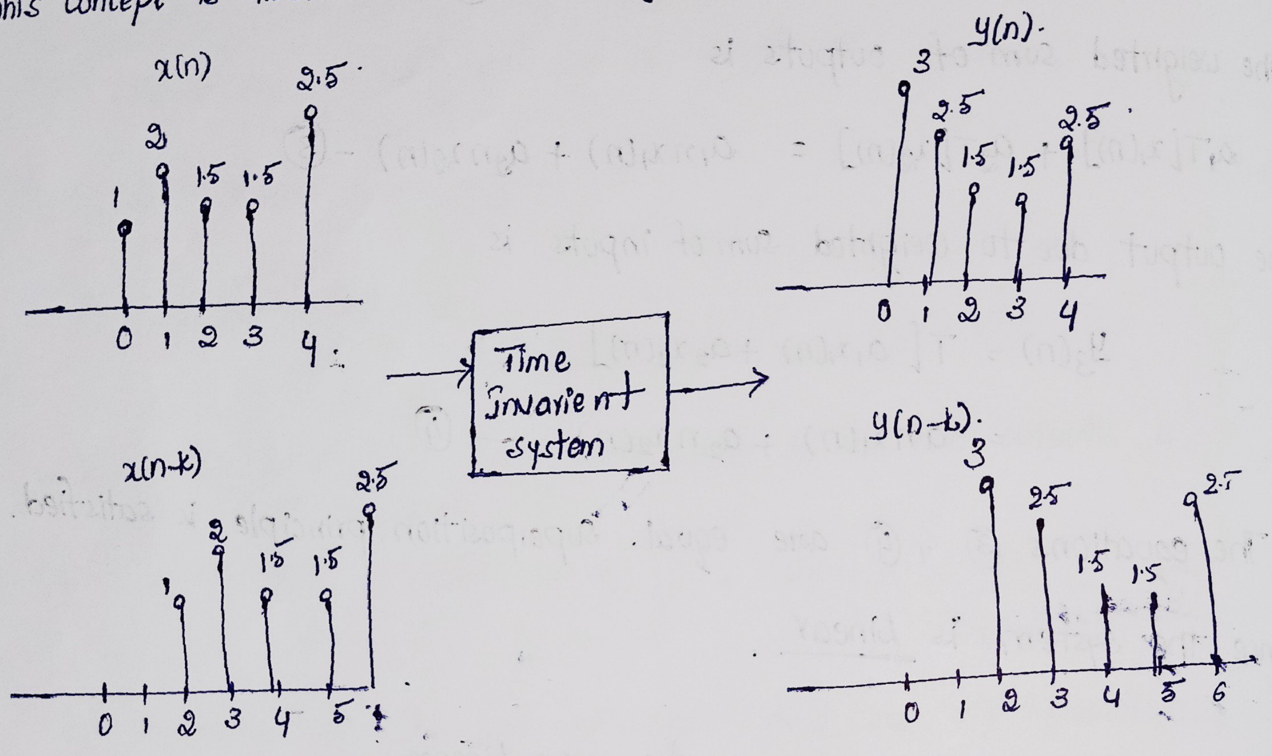


Fig: Time Invarient system.

To test if any given system is time - invarient, first apply an arbitrary sequence $x(n)$ and find $y(n)$. Now delay the input sequences by 'k' samples and find output sequence, denote it is

$$y(n,k) = T[x(n-k)]$$

Delay the output sequence by 'k' samples, denote it as $y(n-k)$ if

$$y(n,k) = y(n-k)$$

for all possible value of 'k', the system is time invarient.

On the otherhand, if the output

$y(n, k) \neq y(n-k)$ even for one value of 'k', the system is Time-Varient.

A Linear Time-Invarient (LTI) discrete-time system satisfies both the Linearity and the Time-Invariance properties.

→ Determine if the following systems are time-Invarient or Time-Varient

(i) $y(n) = x(n) + x(n-1)$

Given $y(n) = T[x(n)] = x(n) + x(n-1)$

If the input is delayed by 'k' units in time, We have

$$y(n,k) = T[x(n-k)]$$

$$= x(n-k) + x(n-k-1) \quad — ①$$

If we delay the output by 'k' units in time then

$$y(n-k) = x(n-k) + x(n-k-1) \quad — ②$$

Here eq ① & ② are equal then the system is <u>Time-Invarient</u>

$$y(n,k) = y(n-k)$$

(ii) $y(n) = x(-n)$

Given $y(n) = T[x(n)] = x(-n)$

If the input is delayed by 'k' units in time and applied to the system. We have

$$y(n,k) = T[x(n-k)] = \cancel{x(-(n+k))}\ x(-n-k) \quad — ①$$

If the output is delayed by 'k' samples

$$y(n-k) = x(-(n-k)) = x(-n+k) \quad — ②$$

From eq ① & ②

$$y(n,k) \neq y(n-k)$$

Therefore, the system is <u>Time-Varient</u>

(iii) $y(n) = x\left(\frac{n}{2}\right)$

$$y(n,k) = T[x(n)] = x\left(\frac{n}{2}-k\right) \quad -①$$

$$y(n-k) = x\left(\frac{n-k}{2}\right) \quad -②$$

$$y(n,k) \neq y(n-k)$$

so, the given system is Time-varient.

(iv) $y(n) = n.x^2(n)$

$$y(n,k) = T[x(n)] = n[x(n-k)]^2 \quad -①$$

$$y(n-k) = (n-k)[x(n-k)]^2 \quad -②$$

From eq ① & ②

$$y(n,k) \neq y(n-k)$$

So the given system is Time-varient.

→ **Representation of an Arbitrary sequence.**

Any arbitary sequence $x(n)$ can be represented in terms of delayed and scaled impulse sequence $\delta(n)$.

Let $x(n)$ be an infinite sequence as shown in figure 1(a)

The sample $x(0)$ can be obtained by multiplying $x(0)$, the magnitude, with unit impulse $\delta(n)$ as shown in Fig 1(c).

i.e
$$x(0)\delta(n) = \begin{cases} x(0) & \text{for } n=0 \\ 0 & \text{for } n\neq 0 \end{cases}$$

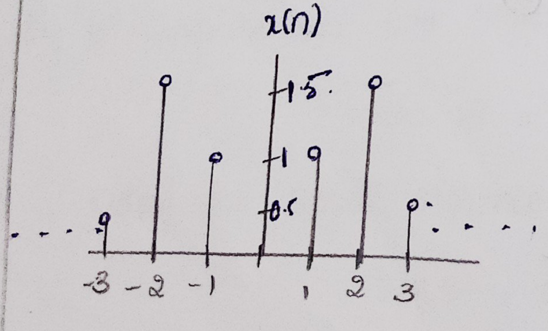


Fig 1(a)

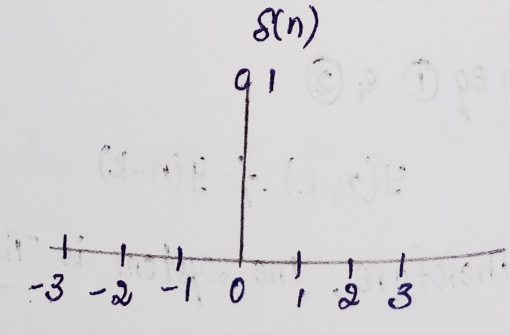


Fig 1(b)

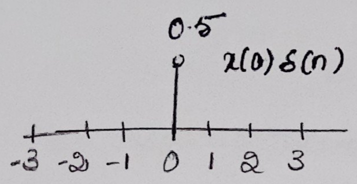


Fig 1(c)

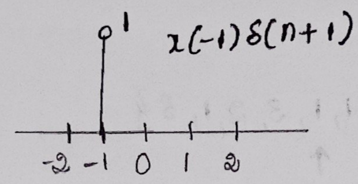


Fig 1(d)

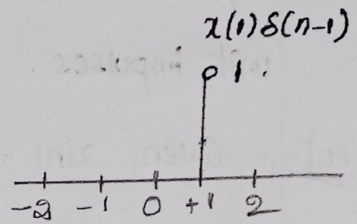


Fig 1(e).

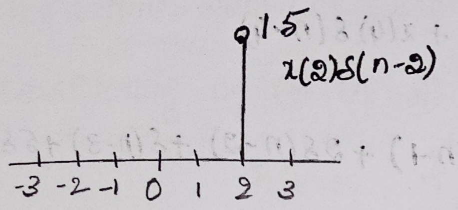


Fig 1(f)

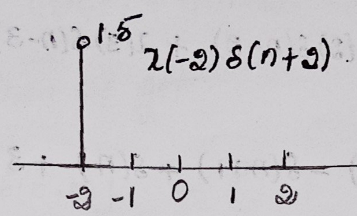


Fig 1(g).

Similarily, the sample $x(-1)$ can be obtained by multiplying $x(-1)$ the magnitude, with one sample advanced unit impulse $\delta(n+1)$ as shown in Fig 1(d).

i.e $\quad x(-1)\delta(n+1) = \begin{cases} x(-1) & \text{for } n=-1 \\ 0 & \text{for } n\neq -1 \end{cases}$

In the same way

$$x(-2)\,\delta(n+2) = \begin{cases} x(-2) & \text{for } n=-2 \\ 0 & \text{for } n\neq -2 \end{cases}$$

$$x(1)\,\delta(n-1) = \begin{cases} x(1) & \text{for } n=1 \\ 0 & \text{for } n\neq 1 \end{cases}$$

$$x(2)\,\delta(n-2) = \begin{cases} x(2) & \text{for } n=2 \\ 0 & \text{for } n\neq 2 \end{cases}$$

The sum of the five sequences shown in Fig 1(c) to 1(g) given by $x(-2)\delta(n+2) + x(-1)\delta(n+1) + x(0)\delta(n) + x(1)\delta(n-1) + x(2)\delta(n-2)$ equals $x(n)$ for $-2 \le n \le 2$. In general, we can write $x(n)$ for $-\infty \le n \le \infty$ as

$$x(n) = \ldots x(-3)\delta(n+3) + x(-2)\delta(n+2) + x(-1)\delta(n+1) + x(0)\delta(n)$$

$$+\, x(1)\delta(n-1) + x(2)\delta(n-2) + x(3)\delta(n-3) + \ldots$$

$= \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$ where $\delta(n-k)$ is unity for $n=k$ and zero for all other terms.

→ Represent the sequence $x(n) = \{4, 2, -1, \underset{\uparrow}{1}, 3, 2, 1, 5\}$ as sum of shifted unit impulses.

<u>sol</u>. Given $x(n) = \{4, 2, -1, \underset{\uparrow}{1}, 3, 2, 1, 5\}$

$n = -3, -2, -1, 0, 1, 2, 3, 4$

$$x(n) = x(-3)\delta(n+3) + x(-2)\delta(n+2) + x(-1)\delta(n+1) + x(0)\delta(n)$$
$$+ x(1)\delta(n-1) + x(2)\delta(n-2) + x(3)\delta(n-3) + x(4)\delta(n-4)$$

$$= 4\delta(n+3) + 2\delta(n+2) - \delta(n+1) + \delta(n) + 3\delta(n-1) + 2\delta(n-2) + \delta(n-3) + 5\delta(n-4)$$

→ <u>Impulse Response and Convolution sum</u>.

A discrete-time system performs an operation on an input signal based on a predefined criteria to produce a modified output signal. The input signal $x(n)$ is the system excitation and $y(n)$ is the system response. This transform operation is shown in Fig

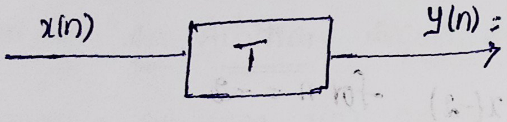


Fig : A Discrete-time System representation.

If the input to the system is a unit impulse i.e $x(n) = \delta(n)$ then the output of the system is known as impulse response denoted by $h(n)$ where

$$h(n) = T[\delta(n)] \quad \text{--- } ①$$

We know that any arbitrary sequence $x(n)$ can be represented as a weighted sum of discrete impulses. Now the system response is given by

$$y(n) = T[x(n)] = T\left[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right] \quad \text{--- } ②$$

For a linear system the eq ② reduces to

$$y(n) = \sum_{k=-\infty}^{\infty} x(k) T[\delta(n-k)] \quad -③$$

The response to the shifted impulse sequence can be denoted by $h(n,k)$ defined as

$$h(n,k) = T[\delta(n-k)] \quad -④$$

For a time-invarient system $h(n,k) = h(n-k)$ $-⑤$

substituting eq ⑤ in ④ then we get

$$h(n-k) = T[\delta(n-k)] \quad -⑥$$

Substitute the eq ⑥ in ③ then we get

$$y(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k) \quad -⑦$$

For a Linear time-Invarient system, if the input sequence $x(n)$ and impulse response $h(n)$ are given, we can find the output $y(n)$ by using the equation

$$y(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$

which is known as Convolution sum and can be represented as

$$y(n) = x(n) * h(n)$$

where $*$ denotes the Convolution operation.

Properties of Convolution

1. Commutative Law : $x(n) * h(n) = h(n) * x(n)$
2. Associative Law : $[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]$
3. Distributive Law : $x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n)$

Procedure for Convolution Sum

step1: choose an initial value of n, the starting time for evaluating the output sequence $y(n)$. If $x(n)$ starts at $n = n_1$ and $h(n)$ starts at $n = n_2$ then $n = n_1 + n_2$ is a good choice.

step2: Express both sequences in terms of index 'k'

step3: Fold $h(k)$ about $k=0$ to obtain $h(-k)$ and shift by 'n' to the right if 'n' is positive and left if 'n' is negative to obtain $h(n-k)$

step4: Multiply the two sequences $x(k)$ and $h(n-k)$ to right, by one sample, ~~and do step4.~~ element by element and sum up the products to get $y(n)$

step5: Increment the index n, shift the sequence $h(n-k)$ to right by one sample and do step 4.

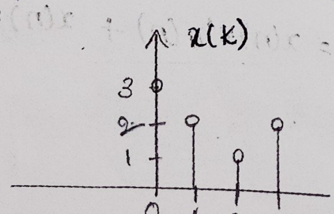
step6: Repeat step 5 until the sum of products is zero for all the remaining values of 'n'

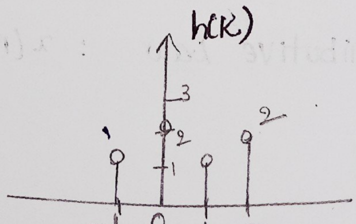
→ Determine the convolution sum of two sequences

$x(n) = \{3, 2, 1, 2\}$; $h(n) = \{1, \underset{\uparrow}{2}, 1, 2\}$

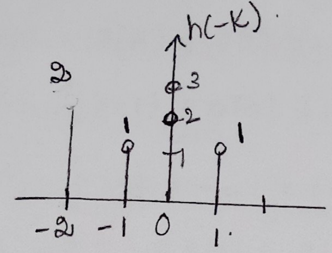
Sol. ① The sequence $x(n)$ starts at $n=0$ and $h(n)$ starts at $n_2 = -1$.
Therefore the starting time for evaluating the output sequence $y(n)$ is $n = n_1 + n_2 = 0 + (-1) = -1$.

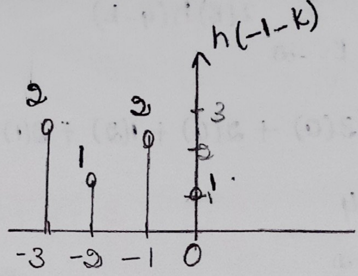
② Express both the sequences in terms of the index 'k'.





step3 : Fold $h(k)$ about $k=0$ to obtain $h(-k)$





We know that $y(-1) = \sum_{k=-\infty}^{\infty} x(k)h(-1-k)$.

Multiply the two sequences $x(k)$ and $h(-1-k)$ element by element and sum of products

$$y(-1) = 0(2) + 0(1) + 0(2) + 3(1) + 2(0) + 1(0) + 2(0) = 3$$

Increment the index by 1, shift the sequence to right to obtain $h(-k)$ and multiply the two sequences $x(k)$ and $h(-k)$ element by element and sum the products.

$$y(0) = \sum_{k=-\infty}^{\infty} x(k)h(-k)$$

$$= 0(2) + 0(1) + 3(2) + 2(1) + 1(0) + 2(0) = 0+0+6+2+0+0 = 8.$$

$$y(1) = \sum_{k=-\infty}^{\infty} x(k)h(1-k)$$

$$= 0(3) + 3(1) + 2(2) + 1(1) + 2(0)$$

$$= 8.$$

$$y(2) = \sum_{k=-\infty}^{\infty} x(k)h(2-k)$$

$$= 3(2) + 2(1) + 1(2) + 2(1)$$

$$= 6+2+2+2 = 12$$

$$y(3) = \sum_{k=-\infty}^{\infty} x(k)h(3-k)$$

$$= 3(0) + 2(2) + 1(1) + 2(2)$$

$$= 9$$

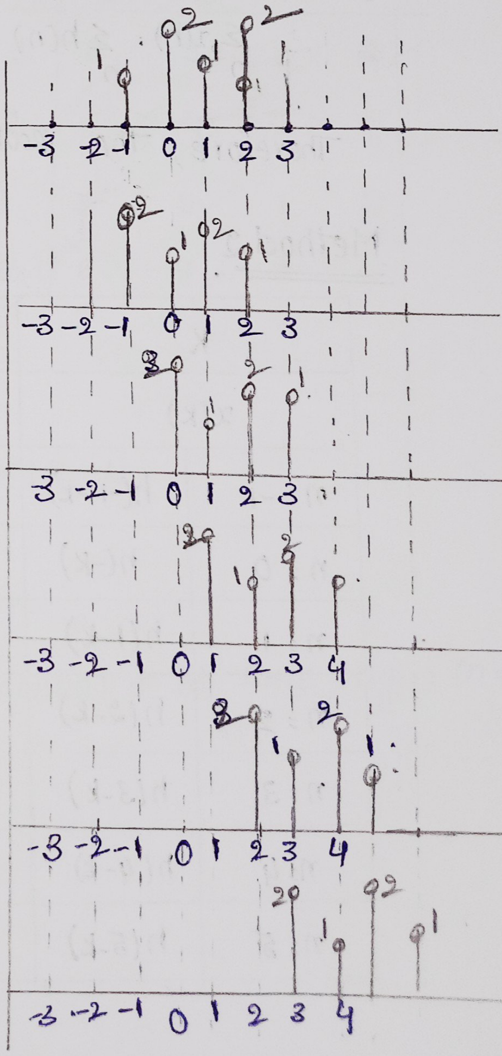
n = 0

n = 2

n = 3

n = 4

n = 5



$$y(4) = \sum_{k=-\infty}^{\infty} x(k)h(4-k)$$

$$= 3(0) + 2(0) + 1(2) + 2(1) + 0(2)$$

$$= 4$$

$$y(5) = \sum_{k=-\infty}^{\infty} x(k)h(5-k)$$

$$= 3(0) + 2(0) + 1(0) + 2(2) + 0(1) + 0(2) + 0(1)$$

$$= 4$$

$$\therefore y(n) = \{3, \underset{\uparrow}{8}, 8; 12, 9, 4, 4\}$$

To check the correctness of the result sum all the samples in $x(n)$ and mortiply by the sum of all samples in $h(n)$. This value must be equal to sum of all samples $y(n)$.

In the given problem $\sum_n x(n) = 8$, $\sum_n h(n) = 6$, $\sum_n y(n) = 48$.

$\therefore \sum_n x(n) \cdot \sum_n h(n) = 8 \times 6 = 48 = \sum_n y(n)$ Proved.

Therefore, the result is correct.

Method 2.

K		-4	-3	-2	-1	0	1	2	3	4	5	6
$x(k)$						3	2	1	2			
$n=-1$	$h(-1-k)$		2	1	2	1						
$n=0$	$h(-k)$			2	1	2	1					
$n=1$	$h(1-k)$				2	1	2	1				
$n=2$	$h(2-k)$					2	1	2	1			
$n=3$	$h(3-k)$						2	1	2	1		
$n=4$	$h(4-k)$							2	1	2	1	
$n=5$	$h(5-k)$								2	1	2	1

$y(-1) = 3(1) = 3$

$y(0) = 3(2) + 2(1) = 6+2 = 8$

$y(1) = 3(1) + 2(2) + 1(1) = 3+4+1 = 8$

$y(2) = 3(2) + 2(1) + 1(2) + 2(1) = 6+2+2+2 = 12$

$y(3) = 2(2) + 1(1) + 2(2) = 4+1+4 = 9$

$y(4) = 1(2) + 2(1) = 2+2 = 4$

$y(5) = 2(2) = 4.$

$\therefore y(n) = \{3, \underset{\uparrow}{8}, 8, 12, 9, 4, 4\}$

Method 3:

Given $x(n) = \{3, 2, 1, 2\}$, $h(n) = \{1, 2, 1, 2\}$

Step 1: Write down the sequence $x(n)$ and $h(n)$ as shown

Step 2: Multiply each and every sample in $h(n)$ by the samples of $x(n)$ and tabulate the values

Step 3: Divide the elements in the table by drawing diagonal lines as shown

Step 4: starting from the left, sum all the elements in each strip and write down in the same order

		$x(n)$: 3	2	1	2
$h(n)$	1	3	2	1	2
	2	6	4	2	4
	1	3	2	1	2
	2	6	4	2	4

Step 5:

$3,\ 6+2,\ 3+4+1,\ 6+2+2+2,\ 4+1+4,\ 2+2,\ 4$

$3,\ 8,\ 8,\ 12,\ 9,\ 4,\ 4$

Step 5: The starting value of $n = -1$ mark the symbol $\uparrow$ at time origin $(n=0)$

$y(n) = \{3, \underset{\uparrow}{8}, 8, 12, 9, 4, 4\}$

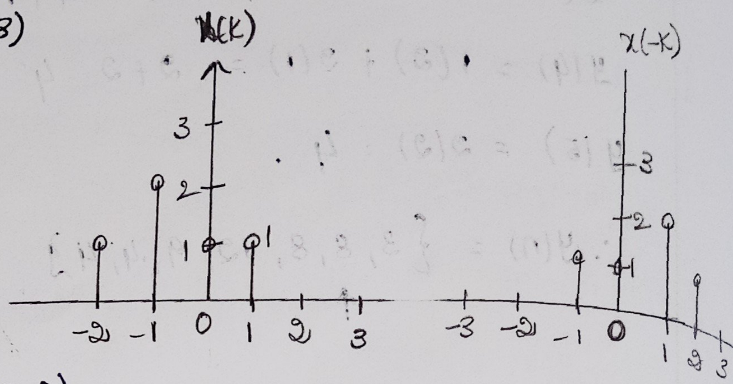
→ Find the Convolution of the signals

$x(n) = 1 \quad n = -2, 0, 1$

$\quad\;\; = 2 \quad n = -1$

$\quad\;\; = 0 \quad$ elsewhere

$h(n) = \delta(n) - \delta(n-1) + \delta(n-2) - \delta(n-3)$



Sol. Given $x(n) = 1$ for $n = -2, 0, 1$

$\quad = 2$ for $n = -1$

$\quad = 0$ elsewhere

$h(n) = \delta(n) - \delta(n-1) + \delta(n-2) - \delta(n-3)$

$x(n) = \{1, 2, 1, 1\}$; $h(n) = \{1, -1, 1, -1\}$
$\quad\;\;\uparrow \qquad\qquad\qquad\qquad\qquad \uparrow$

$n_1 = -2, \; n_2 = 0 \Rightarrow n = n_1 + n_2 = -2 + 0 = -2$

The Length of the sequence $y(n) = N_1 + N_2 - 1$ (or)

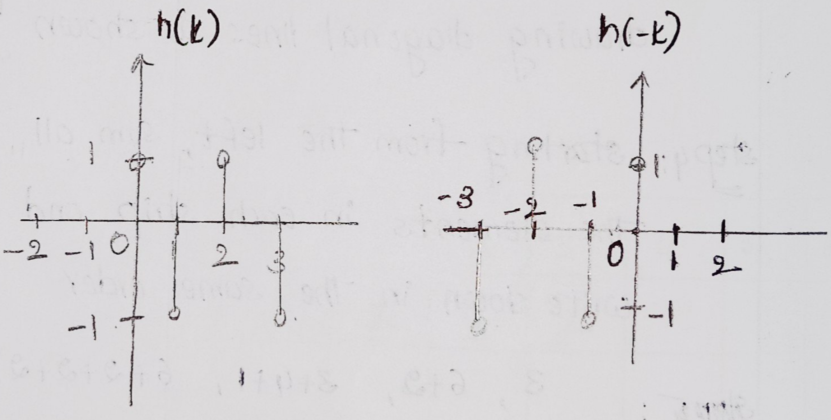
$= 4 + 4 - 1 = 7$

For $n = -2$

$$y(-2) = \sum_{k=-\infty}^{\infty} x(k)\,h(-2-k)$$

$= 0(-1) + 0(1) + 0(-1) + 1(1) + 2(0) + 1(0) + 1(0)$

$= 0 + 0 + 0 + 1 + 0 + 0 + 0 = 1.$



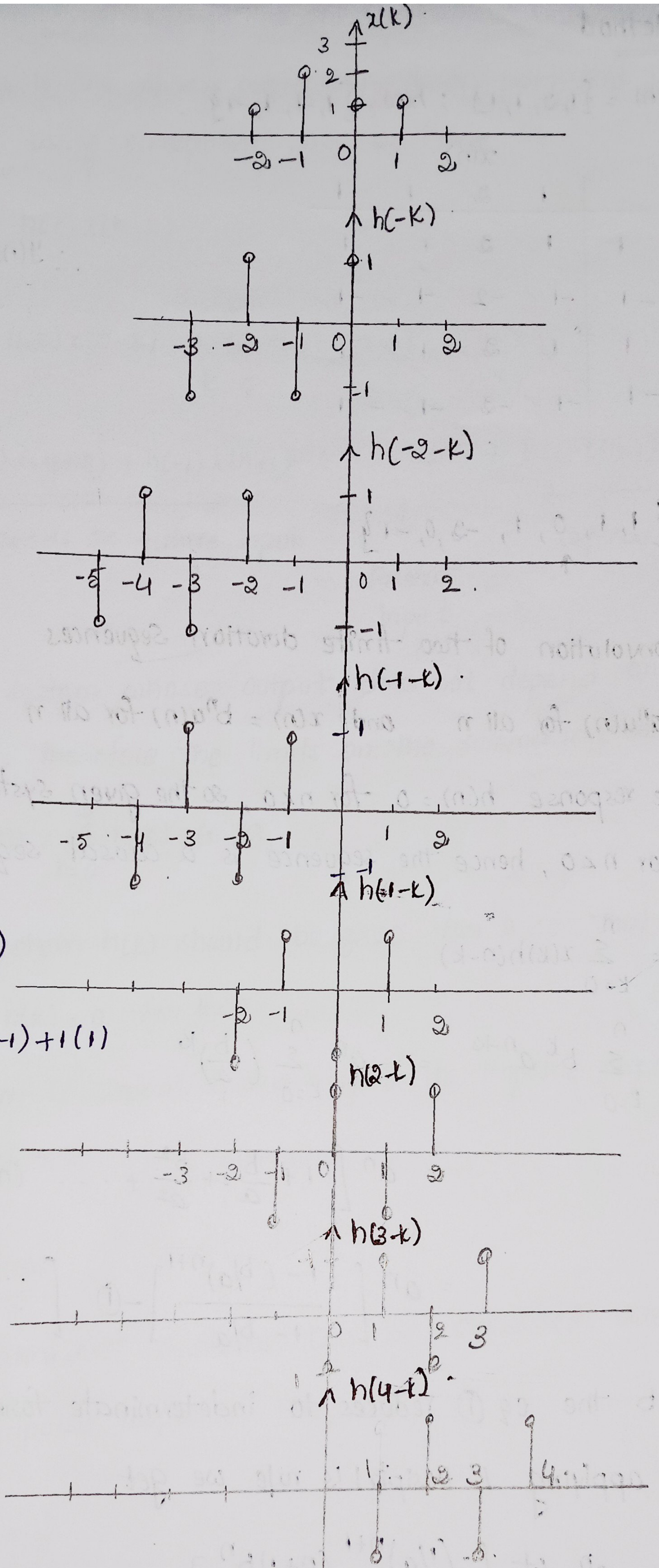
For $n = -1$

$$y(-1) = \sum_{k=-\infty}^{\infty} x(k)\,h(-1-k)$$

$= 0(-1) + 0(1) + 1(-1) + 2(1) = 0 + 0 + (-1) + 2 = 1$

For $n = 0$

$$y(0) = \sum_{k=-\infty}^{\infty} x(k)\,h(-k) = 0(-1) + 1(1) + 2(-1) + 1(1) = 0 + 1 - 2 + 1 = 0$$



For n = 1

$$y(1) = \sum_{k=-\infty}^{\infty} x(k)\, h(1-k)$$

$= 1(-1) + 2(1) + 1(-1) + 1(1)$

$= -1 + 2 - 1 + 1$

$= \underline{1}$

For n = 2

$$y(2) = \sum_{k=-\infty}^{\infty} x(k)\, h(2-k)$$

$= 2(-1) + 1(1) + 1(-1)$

$= -2 + 1 - 1 = \underline{\underline{-2}}$

For n = 3

$y(3) = 1(-1) + 1(1)$

$= 0$

For n = 4

$y(4) = 1(-1) = \underline{-1}$

$$\therefore\ y(n) = \{1, 1, \underset{\uparrow}{0}, 1, -2, 0, -1\}$$

Another Method

Given $x(n) = \{1, 2, 1, 1\}$; $h(n) = \{1, -1, 1, -1\}$.

		$x(n)$			
		1	2	1	1
$h(n)$	1	1	2	1	1
	-1	-1	-2	-1	-1
	1	1	2	1	1
	-1	-1	-2	-1	-1

$\therefore y(n) = \{1,\ 2-1,\ 1-2+1,\ 1-1+2-1,\ -1+1-2,\ 1-1,\ -1\}$

$$y(n) = \{1, 1, \underset{\uparrow}{0}, 1, -2, 0, -1\}$$

→ Find the convolution of two finite duration sequences

$h(n) = a^n u(n)$ for all n and $x(n) = b^n u(n)$ for all n

Sol. The impulse response $h(n) = 0$ for $n < 0$, so the given system is causal and $x(n) = 0$ for $n < 0$, hence the sequence is a causal sequence.

$$y(n) = \sum_{k=0}^{n} x(k)h(n-k)$$

$$= \sum_{k=0}^{n} b^k a^{n-k} = a^n \cdot \sum_{k=0}^{n} \left(\frac{b}{a}\right)^k$$

$$= a^n \left[1 + \frac{b}{a} + \frac{b^2}{a^2} + \dots (n+1) \text{ terms}\right]$$

$$= a^n \left[\frac{1 - (b/a)^{n+1}}{1 - b/a}\right] \quad \text{—(1)} \quad \left[\because \sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}\right]$$

When $a = b$ the eq (1) reduces to indeterminate form.

Therefore, applying L' Hospital's rule we get

$$y(n) = a^n \cdot \underset{b \to a}{Lt} \left[\frac{-(1/a)^{n+1} (n+1) b^n}{(-1/a)}\right]$$

$$= a^n \underset{b \to a}{Lt} (n+1) \left(\frac{b}{a}\right)^n = a^n (n+1)$$

→ Causality

Causal system is one whose output depends on past and present values of the input. Using convolution sum, we have

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\,x(n-k)$$

$$= \sum_{k=-\infty}^{-1} h(k)\,x(n-k) + \sum_{k=0}^{\infty} h(k)\,x(n-k)$$

$$= \ldots \underbrace{h(-2)\,x(n+2) + h(-1)\,x(n+1)}_{\text{depends on future inputs}} + \underbrace{h(0)\,x(n)}_{\text{Present input}} + \underbrace{h(1)\,x(n-1) + \ldots}_{\text{depends on past inputs}}$$

For a causal system whose output does not depend on the future values of the input, therefore the limits on the summation change as

$$y(n) = \sum_{k=0}^{\infty} h(k)\,x(n-k)$$

For a causal system $h(k)$ should be zero for $k<0$. That is

$$h(k) = 0 \quad \text{for } k<0$$

∴ A LTI system is causal if and only if its impulse response is zero for negative values of 'n'

→ FIR and IIR systems

Linear Time-Invarient systems can be classified according to the type of impulse response. They are

1. FIR system
2. IIR system

FIR System : If the impulse response of the system is of finite duration, then the system is called a Finite-Impulse Response (FIR system).

Example of FIR system

$$h(n) = \begin{cases} 1 & \text{for } n = -1, 2 \\ 2 & \text{for } n = 1 \\ 3 & \text{for } n = 0, 3 \\ 0 & \text{otherwise.} \end{cases}$$

IIR system : An Infinite Impulse Response (IIR) system has an impulse response for infinite duration.

Example of an IIR system is.

$$h(n) = a^n u(n)$$

→ **Stable and unstable systems**

An LTI system is stable if it produces a bounded output sequence for every bounded input sequence.

If some bounded input sequence $x(n)$, the output is unbounded (infinite) then the system is called unstable system.

Let $x(n)$ be a bounded input sequence, $h(n)$ be the impulse response of the system and $y(n)$ be the output sequence. Taking magnitude of the output

We have $$|y(n)| = \left|\sum_{k=-\infty}^{\infty} h(k)x(n-k)\right|$$

We know that the magnitude of the sum of terms is less than or equal to the sum of magnitudes, hence

$$|y(n)| \le \sum_{k=-\infty}^{\infty} |h(k)||x(n-k)|$$

The bounded value of the input is equal to M, the above equation become

$$|y(n)| \le M \sum_{k=-\infty}^{\infty} h(k)$$

The above condition will be satisfied when

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

so, the necessary and sufficient condition for stability is

$$\boxed{\sum_{n=-\infty}^{\infty} |h(n)| < \infty}$$

→ Test the stability of the system whose impulse response $h(n) = \left(\frac{1}{2}\right)^n u(n)$

Sol The neccessary and sufficient condition for stability is

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

Given $h(n) = (1/2)^n u(n)$

$$\sum_{n=-\infty}^{\infty} \left|\left(\frac{1}{2}\right)^n u(n)\right| = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

$$= 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \ldots \infty \qquad \left[\because 1 + a + a^2 + \ldots \infty = \frac{1}{1-a}\right]$$

$$= \frac{1}{1-\frac{1}{2}} = 2 < \infty$$

Hence the system is stable.

→ Inverse system and Deconvolution

Inverse system

A system is said to be invertible if the input to the system can be recovered from the output. That is, when the original system is cascaded with its inverse system, the output $w(n)$ is equal to the input $x(n)$ of the original system. This concept illustrated in below figure

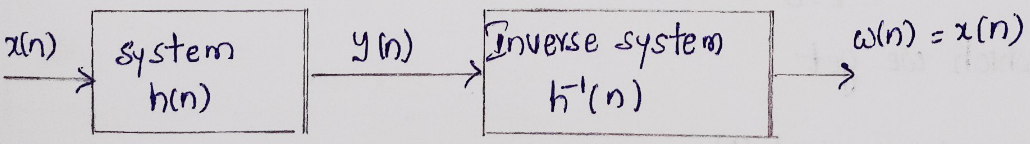


Fig: Cascading of the system and its inverse

For example if $y(n) = ax(n)$

Then $w(n) = \frac{1}{a} y(n)$

If the impulse response of the original system is $h(n)$, then the impulse response of inverse system is $h^{-1}(n)$.

$$y(n) = x(n) * h(n)$$

&

$$w(n) = y(n) * h^{-1}(n)$$

$$= [x(n) * h(n)] * h^{-1}(n)$$

$$= x(n) * [h(n) * h^{-1}(n)] \qquad [\because h(n) * h^{-1}(n) = \delta(n)].$$

$$= x(n)$$

→ Deconvolution

In certain applications like measurements, the knowledge of the impulse response $h(n)$ and output $y(n)$ may be known ahead and we may have to find the input applied to the system.

For example the system may be a temperature tranducer or a Pressure transducer. Then we can find the input to the system by deconvolving $y(n)$ and $h(n)$.

The process of recovering $x(n)$ from $x(n) * h(n)$ is known as deconvolution. We have

$$y(n) = \sum_{k=-\infty}^{\infty} x(k) h(n-k)$$

Assuming $y(n)$ and $h(n)$ are one sided sequence, We have

$$y(n) = \sum_{k=0}^{\infty} x(k) h(n-k)$$

From which we get

$$y(0) = x(0) h(0)$$
$$y(1) = h(1) x(0) + h(0) x(1)$$

$y(2) = h(2)x(0) + h(1)x(1) + h(0)x(2)$

$\vdots$

In Matrix form

$$\begin{bmatrix} y(0) \\ y(1) \\ y(2) \\ \vdots \\ \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} h(0) & 0 & 0 & \cdots & 0 \\ h(1) & h(0) & 0 & \cdots & 0 \\ h(2) & h(1) & h(0) & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \\ \vdots & \vdots & \vdots & & \vdots \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ \vdots \end{bmatrix}$$

$$Y = H \quad X$$

$$Y = HX$$

$$X = H^{-1}Y$$

From the matrix, we get

$y(0) = h(0)x(0) \Longrightarrow x(0) = \dfrac{y(0)}{h(0)}$

$y(1) = h(1)x(0) + h(0)x(1)$

$$x(1) = \frac{y(1) - h(1)x(0)}{h(0)}$$

similarily $y(2) = h(2)x(0) + h(1)x(1) + h(0)x(2)$

$$x(2) = \frac{y(2) - h(1)x(1) - h(2)x(0)}{h(0)}$$

In General

$$\boxed{x(n) = \frac{y(n) - \sum_{k=0}^{n-1} x(k)h(n-k)}{h(0)}}$$

→ What is the input signal $x(n)$ that will generate the output sequence $y(n) = \{1, 5, 10, 11, 8, 4, 1\}$ for a system with impulse response $h(n) = \{1, 2, 1\}$

Sol. Given $y(n) = \{1, 5, 10, 11, 8, 4, 1\}$ and $h(n) = \{1, 2, 1\}$.

The no. of samples in output is $N_1 + N_2 - 1 = 7$.

The no. of samples in impulse response is $N_2 = 3$.

Therefore $N_1 = 5$. That is $x(n)$ have five samples.

From the equation

$$x(n) = \frac{y(n) - \sum_{k=0}^{n-1} x(k)h(n-k)}{h(0)}$$

For $\underline{\underline{n=0}}$

$$x(0) = \frac{y(0)}{h(0)} = \frac{1}{1} = 1.$$

For $\underline{\underline{n=1}}$

$$x(1) = \frac{y(1) - [x(0)h(1)]}{h(0)}$$

$$= \frac{5 - (1 \times 2)}{1} = \frac{5-2}{1} = 3$$

For $\underline{\underline{n=2}}$

$$x(2) = \frac{y(2) - [x(0)h(2) + x(1)h(1)]}{h(0)}$$

$$= \frac{10 - 1(1) - 3(2)}{1} = 3.$$

For $n=3$

$$x(3) = \left. y(3) - \sum_{k=0}^{2} x(k)\,h(3-k) \right/ h(0)$$

$$= y(3) - x(0)h(3) - x(1)h(2) - x(2)h(1) \big/ h(0)$$

$$= 11 - 1(0) - 3(1) - 3(2)$$

$$= 11 - 0 - 3 - 6 = 2$$

For $n=4$

$$x(4) = \left. y(4) - \sum_{k=0}^{3} x(k)\,h(4-k) \right/ h(0)$$

$$= y(4) - x(0)h(4) - x(1)h(3) - x(2)h(2) - x(3)h(1) \big/ h(0)$$

$$= 8 - 1(0) - 3(0) - 3(1) - 2(2) = 1.$$

Therefore $x(n) = \{1, 3, 3, 2, 1\}$.

Time Response Analysis of Discrete-time systems

In this section, we will study about the time-domain behaviour of Linear time-invariant discrete-time system for different standard input signals.

There are two basic methods for analysing the response of a linear system to a given input signal.

The first method is finding response $y(n)$ for any input signal $x(n)$ using convolution sum if impulse response $h(n)$ is known.

The second method is based on the direct solution of the difference equation representing the system.

The general form of difference equation of an N^{th} order Linear-time Invarient discrete-time (LTI-DT) system is

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

where a_k & b_k are constants.

The response of any discrete-time system can be composed as

Total response = zero state response + zero input response.

The zero state response of the system is due to the input alone when the initial state of the system is zero. That is the system is initially relaxed at time $n=0$.

On the otherhand, the zero input response depends only on the initial state of the system that is the input is zero.

For an example, Let us consider a first order discrete-time system with difference equation

$$y(n) = ay(n-1) + x(n)$$

where $x(n)$ & $y(n)$ are input and output respectively. Let the input sequence is zero for $n<0$ and the initial condition of $y(n)$ for $n=-1$ exists. That is $y(-1) \neq 0$. The successive values of $y(n)$ for $n>0$ are as follows.

For $n=0$ $\Rightarrow$ $y(0) = ay(-1) + x(0)$

$$y(1) = ay(0) + x(1)$$
$$= a[ay(-1) + x(0)] + x(1)$$
$$= a^2 y(-1) + ax(0) + x(1)$$

$$y(2) = ay(1) + x(2)$$

$$= a[a^2y(-1) + ax(0) + x(1)] + x(2)$$

$$= a^3y(-1) + a^2x(0) + ax(1) + x(2)$$

For any n

$$y(n) = a^{n+1}y(-1) + a^n x(0) + a^{n-1}x(1) + \cdots + x(n)$$

$$y(n) = a^{n+1}y(-1) + \sum_{k=0}^{n} a^k x(n-k) \quad \text{for } n \geq 0$$

The response $y(n)$ includes two parts. The first part depends on the initial condition of the system and the second term depends on the input.

When $y(-1) = 0$, the output $y(n)$ depends only on the input applied. Hence $y(n)$ is known as the zero state response or forced response of the system given by

$$y_f(n) = \sum_{k=0}^{n} a^k x(n-k) \quad \text{for } n \geq 0$$

If the system is initially non-relaxed that is $y(-1) \neq 0$ and the input $x(n) = 0$ for all n, the output of the system $y(n)$ depends only on the initial state of the system. Then the response of the system is called the zero input response or natural response and is denoted by

$$y_n(n) = a^{n+1}y(-1) \quad \text{for } n \geq -1$$

The solution of the difference equation can be expressed as sum of two parts given by

$$y(n) = y_h(n) + y_p(n)$$

where $y_h(n)$ is known as homogenous or complimentary solution

$y_p(n)$ is called particular solution

→ Natural Response (zero input response)

The difference equation of a N^{th} order discrete-time system can be written as

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k) \quad -①$$

The natural response $y_n(n)$ is the solution of eq ① with $x(n)=0$.

Therefore for a discrete-time system the natural response is the solution of the homogenous equation

$$\sum_{k=0}^{N} a_k y(n-k) = 0 \quad -②$$

The solution of above equation is of the form

$$y_h(n) = \lambda^n \quad -③$$

Substituting eq ③ in eq ②, then

$$\sum_{k=0}^{N} a_k \lambda^{n-k} = 0 \quad ; \quad a_0 = 1$$

$$\lambda^n + a_1\lambda^{n-1} + \dots + a_{N-1}\lambda^{n-N+1} + a_N\lambda^{n-N} = 0$$

$$\lambda^{n-N}\left[\lambda^N + a_1\lambda^{N-1} + \dots + a_{N-1}\lambda + a_N\right] = 0$$

$$\lambda^N + a_1\lambda^{N-1} + \dots + a_{N-1}\lambda + a_N = 0 \quad -④$$

Now, the N^{th} order characteristic equation can be expressed in the factorized form as

$$(\lambda-\lambda_1)(\lambda-\lambda_2)\dots(\lambda-\lambda_N) = 0 \quad -⑤$$

where $\lambda_1, \lambda_2, \dots \lambda_N$ are roots of characteristic equation.

The nature of the natural response depends on the type of roots: real, imaginary and complex.

The real roots lead to real exponential, imaginary roots to sinusoidal and complex roots to exponentially damped sinusoidal.

Distinct roots.

If the roots $\lambda_1, \lambda_2, \lambda_3 \ldots \lambda_N$ of eq ⑤ are distinct, then it has N solutions $C_1\lambda_1^n$, $C_2\lambda_2^n, \ldots C_N\lambda_N^n$. In that case the general solution is of the form

$$y_h(n) = C_1\lambda_1^n + C_2\lambda_2^n + \ldots + C_N\lambda_N^n$$

where $C_1, C_2 \ldots C_N$ are arbitrary constants. These constants are determined by applying initial conditions

For example if the roots are $\lambda_1 = 2$, $\lambda_2 = 3$ then

$$y_h(n) = C_1(2)^n + C_2(3)^n$$

Repeated roots.

If the roots λ_1 is repeated 'm' times and the remaining (N-m) roots are distinct then the characteristic equation of the system is

$$(\lambda - \lambda_1^m)(\lambda - \lambda_{m+1})(\lambda - \lambda_{m+2}) \ldots (\lambda - \lambda_N) = 0$$

and the general solution is.

$$y_h(n) = (C_1 + C_2 n + C_3 n^2 + \ldots C_m n^{m-1})(\lambda_1)^n$$
$$+ C_{m+1}(\lambda_{m+1})^n + C_{m+2}(\lambda_{m+2})^n + \ldots + C_N\lambda^N.$$

For example if the roots of the characteristic equation are $\lambda_1 = -2$, $\lambda_2 = -2$ and $\lambda_3 = 2$ then the solution is

$$y_h(n) = [C_1 + C_2 n]\,(2)^n + C_3(2)^n$$

Complex roots

If the roots are complex, then we can write

$$\lambda_1 = \lambda = a + jb$$

$$\lambda_2 = \lambda^* = a - jb$$

Then the homogenous solution is of the form

$$y_h(n) = r^n \left[A_1 \cos n\theta + A_2 \sin n\theta \right]$$

where $r = \sqrt{a^2+b^2}$ and A_1 & A_2 are constants.

$$\theta = \tan^{-1}\left(\frac{b}{a}\right)$$

→ **Forced Response (zero state Response)**

The forced response is the solution of the difference equation for the given input when the initial conditions are zero. It consists of two parts, homogenous solution and particular solution.

The homogenous solution can be obtained from the roots of characteristic equation. The particular solution $y_p(n)$ is to satisfy the difference equation for the specific input signal $x(n)$, $n \geq 0$

In other words, $y_p(n)$ is a solution satisfying

$$1 + \sum_{k=1}^{N} a_k y_p(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

The general form of the particular solution for several inputs are given in table 1. From the table we can find that, if the input $x(n) = A \sin \omega n$ then $y_p(n) = C_1 \cos \omega n + C_2 \sin \omega n$ where C_1 and C_2 are obtained by substituting $y_p(n)$ and $x(n)$ in the difference equation.

Table : General form of particular solution for several types of inputs.

$x(n)$ input signal	$y_p(n)$ Particular solution
A (step input)	K
AM^n	KM^n
An^M	$K_0 n^M + K_1 n^{M-1} + \dots + K_M$
$A^n N^M$	$A^n [K_0 n^M + K_1 n^{M-1} + \dots + K_M]$
$A \cos\omega n$	
$A \sin \omega n$	$C_1 \cos\omega n + C_2 \sin \omega n$

Note : A, K, M, k_i, C_1 and C_2 are constants.

P → Find the Natural response of the system described by difference equation

$$y(n) + 2y(n-1) + y(n-2) = x(n) + x(n-1)$$

with initial condition $y(-1) = y(-2) = 1$.

Sol.

Given $y(n) + 2y(n-1) + y(n-2) = x(n) + x(n-1)$ — ①

The homogenous equation can be obtained by equating the input terms to zero. That is

$$y(n) + 2y(n-1) + y(n-2) = 0 \quad \text{— ②}$$

The homogenous solution is of the form

$$y_h(n) = \lambda^n \quad \text{— ③}$$

Substituting eq ③ in eq ② then we get

$$\lambda^n + 2\lambda^{n-1} + \lambda^{n-2} = 0$$

$$\lambda^{n-2}[\lambda^2 + 2\lambda + 1] = 0$$

$$(\lambda + 1)^2 = 0$$

$$\lambda_1 = -1, \lambda_2 = -1 \quad \text{— ④}$$

The roots are repeated, therefore the general form of homogenous solution is

$$y_h(n) = C_1(-1)^n + C_2 n(-1)^n \quad \text{— ⑤}$$

$$y_h(n) = C_1(-1)^n + C_2 n(-1)^n$$

— ⑥

For $n=0 \Rightarrow y(0) = C_1$

$n=1 \Rightarrow y(1) = C_1(-1)^1 + C_2(-1)^1$

$y(1) = -C_1 - C_2$ — ⑦

From the homogenous equation

$n=0 \Rightarrow y(0) + 2y(-1) + y(-2) = 0$ — ⑧

Given initial conditions are $y(-1) = y(-2) = 1$

$y(0) + 2 + 1 = 0$

$y(0) = -3$ — ⑨

$n=1$ $\quad y(1) + 2y(0) + y(-1) = 0$

$y(1) + 2(-3) + 1 = 0$

$y(1) = 5$ — ⑩

Equate eq ⑥ & ⑨ then

$y(0) = C_1 = -3$

Equate eq ⑦ & ⑩ then

$y(1) = 5 = -C_1 - C_2$

$5 = 3 - C_2$

$C_2 = -2$

By substituting the eo values in eq ⑤ then

$y_h(n) = (-3)(-1)^n + (-2)n(-1)^n$ for $n \geq 0$

$= (-3)(-1)^n u(n) + (-2)n(-1)^n u(n)$

P) Find the natural response of the system described by different equation

$$y(n) - 4y(n-1) + 4y(n-2) = x(n) - x(n-1)$$

When the initial conditions are $y(-1) = y(-2) = 1$.

Sol. Given $y(n) - 4y(n-1) + 4y(n-2) = x(n) - x(n-1)$ — ①

The Natural response can be obtained by equating the input terms to zero

$$y(n) - 4y(n-1) + 4y(n-2) = 0 \quad — ②$$

The solution for homogenous equation is

$$y(n) = \lambda^n \quad — ③$$

substitute the eq ③ in eq ② then

$$\lambda^n - 4\lambda^{n-1} + 4\lambda^{n-2} = 0$$

$$\lambda^{n-2}[\lambda^2 - 4\lambda + 4] = 0$$

$$(\lambda - 2)^2 = 0$$

$$\lambda_1 = 2, \quad \lambda_2 = 2 \quad — ④$$

The roots are repeated, therefore the general solution of homogenous equation is

$$y_h(n) = C_1(+\lambda_1)^n + C_2 . n(+\lambda_2)^n$$

$$y_h(n) = C_1(2)^n + C_2 . n(2)^n \quad — ⑤$$

For $n=0$, the eq ⑤ becomes

$$y(0) = C_1 \quad — ⑥$$

For $n=1 \Rightarrow y(1) = C_1(2) + C_2 . 1 . (2)^1$

$$y(1) = 2C_1 + 2C_2 \quad — ⑦$$

Apply the value $n=0$ in eq ② then and apply initial conditions i.e $y(-1) = y(-2) = 1$

$$y(0) - 4y(-1) + 4y(-2) = 0$$

$$y(0) - 4(1) + 4(1) = 0$$

$$y(0) = 0 \quad — ⑧$$

For $n=1$ $\Rightarrow y(1)-4y(0)+4y(-1)=0$

$$y(1)-0+4=0$$

$$y(1)=-4 \quad -⑨$$

Equate the eq ⑥ & ⑧ then

$$y(0)=C_1=0$$

Equate the eq ⑦ & ⑨ then

$$y(1)=-4=2C_1+2C_2$$

$$-4=2C_2$$

$$C_2=-2.$$

Substitute the values of C_1 and C_2 in homogenous solution i.e eq ⑤, then

$$y_h(n)=(-2)\,n\,(2)^n \quad \text{for } n\geq 0$$

$$y_h(n)=-2n\,(2)^n u(n)$$

P → Find the forced response of the system described by the difference equation

$$y(n)+2y(n-1)+y(n-2)=x(n)+x(n-1) \text{ for input } x(n)=\left(\tfrac{1}{2}\right)^n u(n)$$

Sol. Forced response contains homogenous response and particular response

$$y_f(n)=y_h(n)+y_p(n).$$

Homogenous response can be obtained by equating input terms to zero

$$y(n)+2y(n-1)+y(n-2)=0.$$

$$\lambda^n+2\lambda^{n-1}+\lambda^{n-2}=0$$

$$\lambda^{n-2}[\lambda^2+2\lambda+1]=0$$

$$(\lambda+1)^2=0$$

$$\lambda_1=-1,\ \lambda_2=-1$$

Roots are repeated then the general solution of homogenous solution is

$$y_h(n)=C_1(\lambda_1)^n+C_2\cdot n\cdot(\lambda_2)^n$$

$$y_h(n) = C_1(-1)^n + C_2 \cdot n \cdot (-1)^n \quad -①$$

For particular solution, the given input is $x(n) = \left(\frac{1}{2}\right)^n u(n)$.

The given input signal is step signal then the particular solution in the prescribed tabular form is K.

$$y_p(n) = K\left(\frac{1}{2}\right)^n u(n) \quad -②$$

substitute the eq ② in the given difference equation, then

$$K\left(\frac{1}{2}\right)^n u(n) + 2K\left(\frac{1}{2}\right)^{n-1} + K\left(\frac{1}{2}\right)^{n-2} = \left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^{n-1} \quad -③$$

For $n=2$, where none of the terms vanish, we get

$$K\left(\frac{1}{2}\right)^2 + 2K\left(\frac{1}{2}\right) + K = \frac{1}{4} + \frac{1}{2}$$

$$\frac{K}{4} + K + K = \frac{1+2}{4} = \frac{3}{4}$$

$$\frac{K + 8K}{4} = \frac{3}{4}$$

$$9K = 3 \Rightarrow K = \frac{3}{9} = \frac{1}{3} \quad -④$$

Therefore the particular solution is

$$y_p(n) = \frac{1}{3}\left(\frac{1}{2}\right)^n u(n) = \frac{1}{3}\left(\frac{1}{2}\right)^n \quad -⑤$$

∴ The forced response

$$y_f(n) = y_h(n) + y_p(n)$$

$$y_f(n) = C_1(-1)^n + C_2 \cdot n \cdot (-1)^n + \frac{1}{3}\left(\frac{1}{2}\right)^n \quad -⑥$$

For $n=0$

$$y(0) = C_1 + \frac{1}{3} \quad -⑦$$

For $n=1$

$$y(1) = C_1(-1)^1 + C_2(-1)^1 + \frac{1}{6}$$

$$y(1) = -C_1 - C_2 + \frac{1}{6} \quad -⑧$$

From the difference equation

$n=0 \Rightarrow y(0) + 2y(-1) + y(-2) = x(0) + x(-1)$

There is no initial conditions, then $y(-1) = y(-2) = 0$ and

$x(n) = (\frac{1}{2})^n$

$x(0) = (\frac{1}{2})^0 = 1$

$x(-1) = 0$ because of $u(n)$.

$x(1) = (\frac{1}{2})$

$y(0) = 1$ — (9)

For $n=1$ then

$y(1) + 2y(0) + y(-1) = x(1) + x(0)$

$y(1) + 2(1) + 0 = \frac{1}{2} + 1$

$y(1) + 2 = \frac{1+2}{2} = \frac{3}{2}$

$y(1) = \frac{3}{2} - 2 = \frac{3-4}{2} = \frac{-1}{2} \rightarrow$ (10)

Equate eq (9) & (7) then

$y(0) = 1 = C_1 + \frac{1}{3}$

$C_1 = 1 - \frac{1}{3} = \frac{2}{3}$

$C_1 = \frac{2}{3}$

Equate eq (8) & (10) then we get

$y(1) = \frac{-1}{2} = -C_1 - C_2 + \frac{1}{6}$

$-\frac{1}{2} = \frac{-2}{3} - C_2 + \frac{1}{6}$

$C_2 = \frac{-2}{3} + \frac{1}{2} + \frac{1}{6} = \frac{-4+3+1}{6}$

$C_2 = 0$

By substituting C_1 & C_2 values in eq (6) then we get

$y_f(n) = (\frac{2}{3})(-1)^n + 0 + \frac{1}{3}(\frac{1}{2})^n$

$\therefore y_f(n) = (\frac{2}{3})(-1)^n + \frac{1}{3}(\frac{1}{2})^n$

→ Find the forced response of the system described by differential equation

$$y(n)-4y(n-1)+4y(n-2) = x(n)-x(n-1)$$

when the input is $x(n) = (-1)^n u(n)$

Sol: The homogenous or natural response can be obtained by equating input terms to zero

$$y(n)-4y(n-1)+4y(n-2) = 0 \quad -①$$

The solution for homogenous equation is $y(n) = \lambda^n$

$$\lambda^n - 4\lambda^{n-1} + 4\lambda^{n-2} = 0$$

$$\lambda^{n-2}[\lambda^2 - 4\lambda + 4] = 0$$

$$(\lambda-2)^2 = 0$$

$$\lambda_1 = 2, \quad \lambda_2 = 2$$

The roots are repeated then the general solution of homogenous equation is

$$y_h(n) = C_1(2)^n + n\cdot C_2(2)^n \quad -②$$

For particular solution, the given input is step response i.e $(-1)^n u(n)$, then particular solution is $y_p(n) = K(-1)^n u(n) \quad -③$

substitute the eq ③ in the given difference equation ie

$$K(-1)^n - 4K(-1)^{n-1} + 4K(-1)^{n-2} = (-1)^n - (-1)^{n-1}$$

For $n=2$, where none of the terms vanish, we get

$$K(-1)^2 - 4K(-1) + 4(-1)^0 K = (-1)^2 - (-1)$$

$$K + 4K + 4K = 1+1$$

$$9K = 2$$

$$K = \frac{2}{9}$$

Substitute the 'K' value in eq ③, then

$$y_p(n) = \frac{2}{9}(-1)^n u(n) = \frac{2}{9}(-1)^n \quad -④$$

Forced response contains homogenous solution (or) response and particular response

$$y_f(n) = y_h(n) + y_p(n)$$

$$y_f(n) = C_1(2)^n + nC_2(2)^n + \frac{2}{9}(-1)^n \quad -⑤$$

For $\underline{\underline{n=0}}$

$$y(0) = C_1(2)^0 + \frac{2}{9}$$

$$y(0) = C_1 + \frac{2}{9} \quad -⑥$$

For $\underline{\underline{n=1}}$

$$y(1) = C_1(2) + 1 \cdot C_2(2) + \frac{2}{9}(-1)^1$$

$$y(1) = 2C_1 + 2C_2 - \frac{2}{9} \quad -⑦$$

From the difference equation,

$\underline{\underline{n=0}}$

$$y(0) - 4y(-1) + 4y(-2) = x(0) - x(-1)$$

There is no initial conditions are given

$$y(-1) = y(-2) = 0$$

$x(0) = (-1)^0 u(n) = 1$; $x(1) = (-1)^1 u(n) = -1$

$$x(-1) = 0.$$

$$y(0) = 1 \quad -⑧$$

For $\underline{\underline{n=1}}$

$$y(1) - 4y(0) + 4y(-1) = x(1) - x(0)$$

$$y(1) - 4 + 0 = -1 - 1$$

$$y(1) = -2 + 4 = 2$$

$$y(1) = 2 \quad -⑨$$

Equate the eq ⑥ & ⑧ then, we get.

$$y(0) = C_1 + \frac{2}{9} = 1$$

$$C_1 = 1 - \frac{2}{9} = \frac{7}{9}.$$

Equate the eq ⑦ & ⑨ then we get

$$y(1) = 2C_1 + 2C_2 - \frac{2}{9} = 2$$

$$2\left(\frac{7}{9}\right) + 2C_2 - \frac{2}{9} = 2$$

$$\frac{7}{9} + C_2 - \frac{1}{9} = 1$$

$$C_2 + \frac{6}{9} = 1$$

$$C_2 = 1 - \frac{6}{9} = \frac{9-6}{9} = \frac{3}{9} = \frac{1}{3}$$

$$\therefore C_2 = \frac{1}{3}$$

Substitute the values of C_1 & C_2 then we get in eq ⑤

$$y_f(n) = \left(\frac{7}{9}\right)(2)^n + n\left(\frac{1}{3}\right)(2)^n + \frac{2}{9}(-1)^n$$

$$y_f(n) = \left(\frac{7}{9} + \frac{n}{3}\right)(2^n)\,u(n) + \frac{2}{9}(-1)^n u(n)$$

Total Response

The total response is obtained by adding the natural response and forced response

$$y(n) = y_n(n) + y_f(n)$$

If there is no need to find the forced response and natural response separately, the total response can be found in the same way as forced response, by using the actual initial conditions instead of zero initial conditions.

→ Find the response of the system described by the difference equation

$$y(n) + 2y(n-1) + y(n-2) = x(n) + x(n-1).$$

for the input $x(n) = \left(\frac{1}{2}\right)^n u(n)$ with initial conditions $y(-1) = y(-2) = 1$

Sol: The forced response contains homogenous solution and particular solution

$$y_f(n) = y_h(n) + y_p(n) \quad - ①$$

The Homogenous solution can be obtained by equating initial input to zero

$$y(n) + 2y(n-1) + y(n-2) = 0 \quad - ②$$

The general solution of homogenous equation is $y(n) = \lambda^n$

$$\lambda^n + 2\lambda^{n-1} + \lambda^{n-2} = 0$$

$$\lambda^{n-2}[\lambda^2 + 2\lambda + 1] = 0$$

$$(\lambda+1)^2 = 0$$

$$\lambda_1 = -1, \; \lambda_2 = -1$$

The roots are repeated then solution of homogenous equation is

$$y_h(n) = C_1(-1)^n + C_2 \cdot n \cdot (-1)^n \quad - ③$$

Given input is step signal then the particular solution is

$$y_p(n) = K\left(\frac{1}{2}\right)^n u(n) \quad - ④$$

Substitute the particular solution in the given difference equation

$$K\left(\frac{1}{2}\right)^n + 2K\left(\frac{1}{2}\right)^{n-1} + K\left(\frac{1}{2}\right)^{n-2} = \left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^{n-1}$$

For $n=2$, where none of the terms vanish, we get

$$K\left(\frac{1}{2}\right)^2 + 2K\left(\frac{1}{2}\right) + K = \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)$$

$$\frac{K}{4} + K + K = \frac{1}{4} + \frac{1}{2}$$

$$\frac{K}{4} + 2K = \frac{1+2}{4} = \frac{3}{4}$$

$$\frac{K+8K}{4} = \frac{3}{4}$$

$$9K = 3 \Rightarrow K = \frac{1}{3}$$

Substitute the value of K in the given particular equation i.e eq 4

$$y_p(n) = \frac{1}{3}\left(\frac{1}{2}\right)^n \quad — 5$$

Forced response is $y_F(n) = y_h(n) + y_p(n)$

$$y(n) = C_1(-1)^n + C_2 \cdot n \cdot (-1)^n + \frac{1}{3}\left(\frac{1}{2}\right)^n \quad — 6$$

For $n=0$

$$y(0) = C_1 + \frac{1}{3} \quad — 7$$

$n=1$

$$y(1) = C_1(-1) + C_2 \cdot 1 \cdot (-1) + \frac{1}{3} \cdot \frac{1}{2}$$

$$y(1) = -C_1 - C_2 + \frac{1}{6} \quad — 8$$

Substitute $n=0$ in the given difference equation

$$y(0) + 2y(-1) + y(-2) = x(0) + x(-1)$$

$$y(0) + 2(1) + 1 = 1$$

$$y(0) + 3 = 1$$

$$y(0) = -2 \quad — 9$$

$x(n) = \left(\frac{1}{2}\right)^n u(n)$

$x(0) = \left(\frac{1}{2}\right)^0 = 1$

$x(1) = \left(\frac{1}{2}\right) = \frac{1}{2}$

For $n=1$

$$y(1) + 2y(0) + y(-1) = x(1) + x(0)$$

$$y(1) + 2(-2) + 1 = \frac{1}{2} + 1$$

$$y(1) - 4 + 1 = \frac{1}{2} + 1$$

$$y(1) = \frac{1}{2} + 4 = \frac{9}{2} \quad -(10)$$

27
$\frac{15}{12}$

Equate eq (7) & (9)

$$y(0) = C_1 + \frac{1}{3} = -2$$

$$C_1 = -2 - \frac{1}{3} = \frac{-6-1}{3} = -\frac{7}{3}$$

Equate eq (8) & (10)

$$y(1) = -C_1 - C_2 + \frac{1}{6} = \frac{9}{2}$$

$$\frac{7}{3} - C_2 + \frac{1}{6} = \frac{9}{2}$$

$$-C_2 = \frac{9}{2} - \frac{7}{3} - \frac{1}{6}$$

$$= \frac{27 - 14 - 1}{6} = \frac{12}{6}$$

$$C_2 = -2$$

By substituting the values of C_1 & C_2 in eq(6) then the complete response is

$$y(n) = -\frac{7}{3}(-1)^n + (-2)\cdot n\cdot(-1)^n + \frac{1}{3}\left(\frac{1}{2}\right)^n u(n) \text{ for } n\geq 0$$

$$y(n) = -\frac{7}{3}(-1)^n u(n) - 2n(-1)^n u(n) + \frac{1}{3}\left(\frac{1}{2}\right)^n u(n)$$

→ Impulse Response

The general form of difference equation of a N^{th} order system is given by

$$1 + \sum_{k=1}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k) \qquad N \geq M \quad -(1)$$

For input $x(n) = \delta(n)$, we obtain

$$1 + \sum_{k=1}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k \delta(n-k) \quad -(2)$$

For $n > M$, the above equation reduces to homogenous equation

$$\sum_{k=1}^{N} a_k y(n-k) = 0 \; ; \quad a_0 = 1 \quad -(3)$$

We can obtain $h(n)$ by solving eq (3) and imposing the initial conditions to determine the arbitrary constants.

* If $N=M$, we have add an impulse response function to the homogeneous solution

→ Determine the impulse response $h(n)$ for the system described by the Second order difference equation

$$y(n) = 0.6y(n-1) - 0.08y(n-2) + x(n)$$

Sol. Given $y(n) = 0.6y(n-1) - 0.08y(n-2) + x(n)$

We know the total Response,

$$y(n) = y_h(n) + y_p(n) \quad -①$$

For impulse $x(n) = \delta(n)$, the particular solution

$$y_p(n) = 0 \quad -②$$

then eq ① becomes

$$y(n) = y_h(n) \quad -③$$

The homogenous solution can be found by substituting $x(n) = 0$

$$y(n) - 0.6y(n-1) - 0.08y(n-2) = 0 \quad -④$$

The solution for homogenous equation, $y(n) = \lambda^n$

$$\lambda^n - 0.6\lambda^{n-1} - 0.08\lambda^{n-2} = 0$$

$$\lambda^{n-2}[\lambda^2 - 0.6\lambda - 0.08] = 0.$$

$$\lambda^2 - 0.6\lambda - 0.08 = 0$$

The roots of the characteric equation is

$$\lambda_1 = 0.4 \text{ and } \lambda_2 = 0.2. \quad -⑤$$

The roots are distinct then the general solution of homogenous equation is

$$y_h(n) = C_1(\lambda_1)^n + C_2(\lambda_2)^n$$

$$y_h(n) = C_1(0.4)^n + C_2(0.2)^n \quad -⑥$$

For $n=0 \Rightarrow$ $y(0) = C_1 + C_2 \quad -⑦$

For $n=1 \Rightarrow$ $y(1) = 0.4C_1 + C_2(0.2)$

$$= 0.4C_1 + 0.2C_2 \quad -⑧$$

From the difference equation

$n=0 \Rightarrow \quad y(0) = 0.6y(-1) - 0.08y(-2) + x(0)$ $\quad [\because x(0) = \delta(0) = 1]$

$= 0.6(0) - 0.08(0) + 1$

$y(0) = 1 \quad - (9)$

$n=1 \Rightarrow \quad y(1) = 0.6y(0) - 0.08y(-1) + x(1)$

$= 0.6(1) - 0.08(0) + 0$

$y(1) = 0.6 \quad - (10)$

$0.4C_1 + 0.2C_2 = 0.6$
$C_1 + C_2 = 1$

$0.4C_1 + 0.2C_2 = 0.6$
$0.4C_1 + 0.4C_2 = 0.4$
$-0.2C_2 = 0.2$
$C_2 = -1$

Compare the eq (7) & (9) then we get

$y(0) = C_1 + C_2 = 1 \quad - (11)$

Illy compare (or) equate eq (8) & (10) then we get

$y(1) = 0.6 = 0.4C_1 + 0.2C_2 \quad - (12)$

By solving the eq (11) & (12) we get

$C_2 = -1$ and $C_1 + C_2 = 1$

$C_1 - 1 = 1$

$C_1 = 2$

Substitute the value of C_1 and C_2 in eq (6) then the equation is

$y(n) = 2(0.4)^n + (-1)(0.2)^n$ for $n>0$

$y(n) = 2(0.4)^n u(n) - (0.2)^n u(n)$

P. → Determine the impulse response $h(n)$ for the system described by difference equation

$y(n) + y(n-1) - 2y(n-2) = x(n-1) + 2x(n-2)$

Sol: Given

$y(n) + y(n-1) - 2y(n-2) = x(n-1) + 2x(n-2)$

Since $M = N = 2$, the homogenous solution include an impulse term.

The total response is given by

$$y(n) = y_h(n) + y_p(n) \quad -①$$

For input $x(n) = \delta(n)$, the particular solution $y_p(n) = 0$

$$\therefore y(n) = y_h(n) \quad -②$$

The homogenous solution can be found by equating the input terms to zero.

that is

$$y(n) + y(n-1) - 2y(n-2) = 0 \quad -③$$

Let the homogenous solution $y_h(n) = \lambda^n$, substituting this solution in eq ③ then

$$\lambda^n + \lambda^{n-1} - 2\lambda^{n-2} = 0$$

$$\lambda^{n-2}[\lambda^2 + \lambda - 2] = 0$$

Therefore, the roots are 1, -2 and the general form of the solution

to the homogenous equation is

$$y_h(n) = C_1(\lambda_1)^n + C_2(\lambda_2)^n + A\cdot\delta(n)$$

$$= C_1(1)^n + C_2(-2)^n + A\,\delta(n) \quad -④$$

From the difference equation

$\underline{n=0}$ $\Rightarrow$ $y(0) + y(-1) - 2y(-2) = x(-1) + 2x(-2)$

$$y(0) = 0$$

$n=1$ $\Rightarrow$ $y(1) + y(0) - 2y(-1) = x(0) + 2x(-1)$

$$y(1) = 1$$

$n=2$ $\Rightarrow$ $y(2) + y(1) - 2y(0) = x(1) + 2x(0)$

$$y(2) + 1 - 0 = 0 + 2$$

$$y(2) = 1$$

$$\therefore \left.\begin{aligned} y(0) &= 0 \\ y(1) &= 1 \\ y(2) &= 1 \end{aligned}\right\} -⑤$$

substituting $n=0$, $n=1$, and $n=2$ in eq ④, we get

$$\left.\begin{aligned} y(0) &= C_1 + C_2 + A \\ y(1) &= C_1 - 2C_2 \\ y(2) &= C_1 - 4C_2 \end{aligned}\right\} \quad ⑥$$

By comparing $y(1)$ & $y(2)$ the equations are

$$\begin{aligned} C_1 - 2C_2 &= 1 \\ -C_1 \underset{+}{-} 4C_2 &= \underset{-}{1} \\ \hline 2C_2 &= 0 \\ C_2 &= 0 \,; \end{aligned}$$

$$C_1 - 0 = 1$$
$$C_1 = 1$$

$$y(0) = C_1 + C_2 + A$$
$$0 = 1 + 0 + A$$
$$A = -1$$

By substituting C_1, C_2, & A values in eq ④ then

$$y(n) = (1)^n + 0 + (-1)\,\delta(n)$$

$$= u(n) - \delta(n)$$

$$y(n) = u(n-1)$$

∵ From the property of impulse resp

$$\delta(n) = u(n) - u(n-1)$$

$$u(n-1) = u(n) - \delta(n)$$

Previous problem

P. Find the impulse response and step response of a discrete-time Linear time invarient system whose difference equation is given by

$$y(n) = y(n-1) + 0.5y(n-2) + x(n) + x(n-1)$$

Sol Given

$$y(n) = y(n-1) + 0.5y(n-2) + x(n) + x(n-1) \quad - ①$$

For impulse response, the particular solution $y_p(n) = 0$

$$y(n) = y_h(n) \quad - ②$$

The homogenous solution can be obtained by solving the homogenous equation

$$\lambda^n - \lambda^{n-1} + 0.5\lambda^{n-2} = 0$$

$$\lambda^{n-2}(\lambda^2 - \lambda + 0.5) = 0$$

$$\lambda^2 - \lambda + 0.5 = 0 \quad - ③$$

$\lambda^2 - \lambda - 0.5 = 0$

By solving the above equation

$\lambda_1 = 1.366$ and $\lambda_2 = -0.366$

$\therefore$ The roots are distinct then the solution of homogenous equation is

$$y(n) = C_1(\lambda_1)^n + C_2(\lambda_2)^n$$

$$y(n) = C_1(1.366)^n + C_2(-0.366)^n \quad -(4)$$

From the difference equation, we can find

$n=0 \Rightarrow y(0) = y(-1) + 0.5y(-2) + x(0) + x(-1)$

$y(0) = 1$

$n=1 \Rightarrow y(1) = y(0) + 0.5y(-1) + x(1) + x(0)$

$= 1 + 0 + 0 + 1$

$= 2$

$$y(0) = C_1 + C_2$$

$$y(1) = 1.366\,C_1 - 0.366\,C_2$$

By comparing the equations

$$y(0) = 1 = C_1 + C_2$$

$$y(1) = 2 = 1.366\,C_1 - 0.366\,C_2$$

$$C_1 = 1.366$$

$$C_2 = -0.366$$

By substituting the values C_1 & C_2 in eq (4) then

$$y(n) = 1.366(1.366)^n - 0.366(-0.366)^n \quad -(5)$$

Step response

For step input $x(n) = u(n)$, the particular solution $y_p(n) = Ku(n)$.

substituting $x(n)$ and $y_p(n)$ in difference equation

$$Ku(n) = Ku(n-1) + 0.5Ku(n-2) + u(n) + u(n-1) \quad -(6)$$

For $n=2$ where none of the terms vanish we get

$$K = K + 0.5K + 1 + 1$$

$$0.5K = -2 \Rightarrow K = \frac{-20}{5} = -4$$

Substitute the value of 'k' in particular solution

$$y_p(n) = -4u(n) \quad - (7)$$

The total response

$$y(n) = y_h(n) + y_p(n)$$

$$y(n) = C_1(1.366)^n + C_2(-0.366)^n - 4u(n) \quad - (8)$$

For $n=0 \Rightarrow y(0) = C_1 + C_2 - 4$

$n=1 \Rightarrow y(1) = 1.366C_1 - 0.366C_2 - 4$ $\Big\} - (9)$

For step input from the difference equation

$n=0 \Rightarrow y(0) = y(-1) + 0.5y(-2) + x(0) + x(-1)$

$y(0) = 1$

$n=1 \Rightarrow y(1) = y(0) + 0.5y(-1) + x(1) + x(0)$

$= 1 + 0 + 1 + 1$

$y(1) = 3$

$\therefore \left.\begin{array}{l} y(0) = 1 \\ y(1) = 3 \end{array}\right\} - (10)$

By comparing the equations (9) & (10)

$C_1 + C_2 - 4 = 1$; $1.366C_1 - 0.366C_2 - 4 = 3 = y(1)$

$C_1 + C_2 = 5$; $1.366C_1 - 0.366C_2 = 7$

By solving the above equations then

$C_1 = 5.098$, $C_2 = -0.098$

By substituting the C_1 & C_2 in eq (8), then

$y(n) = 5.098(1.366)^n - 0.098(-0.366)^n - 4$ for $n \geq 0$

$y(n) = 5.098(1.366)^n u(n) - 0.098(-0.366)^n u(n) - 4u(n)$

→ Frequency domain representation of Discrete-time signals and systems

The Discrete-time Fourier transform of $x(n)$ is $X(e^{j\omega})$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

The Inverse discrete-time Fourier transform of $X(e^{j\omega})$ is $x(n)$

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} \, d\omega$$

$$\therefore \quad X(e^{j\omega}) = F[x(n)]$$

$$x(n) = F^{-1}[X(e^{j\omega})]$$

$$x(n) \underset{I.F.T}{\overset{F.T}{\rightleftarrows}} X(e^{j\omega})$$

→ Find the Fourier transform of the following

(i) $\delta(n)$ (iii) $\delta(n-k)$ (v) $a^n u(n)$

(ii) $u(n)$ (iv) $u(n-k)$ (vi) $\delta(n+2) - \delta(n-2)$

Sol. (i) Given $x(n) = \delta(n)$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta(n) e^{-j\omega n}$$

$$= e^{-j\omega n}\Big|_{n=0} \qquad \left[\begin{array}{l} \delta(n) = 1 \text{ for } n=0 \\ \quad = 0 \text{ for } n \neq 0 \end{array}\right]$$

$$F[\delta(n)] = 1$$

(ii) Given $x(n) = u(n)$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} u(n) e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} e^{-j\omega n}$$

$$= 1 + e^{-j\omega} + e^{-2j\omega} + \dots \infty$$

$$F[u(n)] = \frac{1}{1-e^{-j\omega}} \qquad \left[\because 1+r+r^2+\dots\infty = \frac{1}{1-r}\right]$$

(iii) Given $x(n) = \delta(n-k)$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta(n-k) e^{-j\omega n}$$

$$= e^{-j\omega k}$$

(iv) Given $u(n-k) = x(n)$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} u(n-k) e^{-j\omega n}$$

$$= \sum_{n=k}^{\infty} e^{-j\omega n}$$

$$= e^{-j\omega k} + e^{-j\omega(k+1)} \dots \infty$$

$$= e^{-j\omega k}\left[1 + e^{-j\omega} + e^{-2j\omega} + \dots \infty\right]$$

$$= e^{-j\omega k} \cdot \frac{1}{1-e^{-j\omega}} = \frac{e^{-j\omega k}}{1-e^{-j\omega}}$$

(v) $a^n u(n)$

Given $x(n) = a^n u(n)$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^n u(n) e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} a^n . e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n$$

$$= 1 + ae^{-j\omega} + (ae^{-j\omega})^2 + \dots \infty$$

$$= \frac{1}{1 - ae^{-j\omega}}$$

(vi) $\delta(n+2) - \delta(n-2)$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} [\delta(n+2) - \delta(n-2)] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \delta(n+2) e^{-j\omega n} - \sum_{n=-\infty}^{\infty} \delta(n-2) e^{-j\omega n}$$

$$= e^{-j\omega(-2)} - e^{-j\omega(2)}$$

$$= e^{j2\omega} - e^{-j2\omega}$$

$$= 2\sin 2\omega \quad \left[\frac{e^{j\theta} - e^{-j\theta}}{2} = \sin\theta\right]$$

→ Evaluate the Fourier transform of the system whose input is unit sample response.

$$h(n) = 1 \text{ for } 0 \le n \le N-1$$
$$= 0 \text{ elsewhere.}$$

sol. The frequency response is given by

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n) e^{-j\omega n}$$

$$= \sum_{n=0}^{N-1} e^{-j\omega n}$$

$$= 1 + e^{-j\omega} + \dots N$$

$\because$ sum of finite terms $= \frac{1-r^N}{1-r}$

$$= \frac{1 - e^{-j\omega N}}{1 - e^{-j\omega}} =$$

$$= \frac{1 - (\cos\omega N - j\sin\omega N)}{1 - (\cos\omega - j\sin\omega)} = \frac{1 - \cos\omega N + j\sin\omega N}{1 - \cos\omega + j\sin\omega}$$

$$= \frac{2\sin^2\frac{\omega N}{2} + j\left(2\sin\frac{\omega N}{2}\cos\frac{\omega N}{2}\right)}{2\sin^2\frac{\omega}{2} + j\left(2\sin\frac{\omega}{2}\cos\frac{\omega}{2}\right)} = \frac{2\sin\frac{\omega N}{2}\left[\sin\frac{\omega N}{2} + j\cos\frac{\omega N}{2}\right]}{2\sin\frac{\omega}{2}\left[\sin\frac{\omega}{2} + j\cos\frac{\omega}{2}\right]} \times \frac{-j}{-j}$$

$$= \frac{2\sin\frac{\omega N}{2}\left[\cos\frac{\omega N}{2} - j\sin\frac{\omega N}{2}\right]}{2\sin\frac{\omega}{2}\left[\cos\frac{\omega}{2} - j\sin\frac{\omega}{2}\right]} = \frac{2\sin\frac{\omega N}{2} \cdot e^{-j\omega N/2}}{2\sin\frac{\omega}{2} \cdot e^{-j\omega/2}}$$

$$= \frac{\sin \frac{\omega N}{2} e^{-j\omega N/2}}{\sin \frac{\omega}{2} \cdot e^{-j\omega/2}}$$

$$= \frac{\sin \frac{\omega N}{2}}{\sin \frac{\omega}{2}} e^{-j(N-1)\omega/2}$$

→ **Transfer function**.

If $H(e^{j\omega})$ is the Fourier transform of the impulse response $h(n)$ and $X(e^{j\omega})$ is the Fourier transform of the input sequence, we can derive the relationship between $Y(e^{j\omega})$, the Fourier transform of output in term of $X(e^{j\omega})$ and $H(e^{j\omega})$.

Any arbitrary sequence can be represented in the form

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{-j\omega}) e^{j\omega n} \, d\omega \quad -(1)$$

where $$X(e^{-j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \quad -(2)$$

We know that $$y(n) = \underbrace{e^{j\omega n}}_{\text{input}} \cdot \underbrace{H(e^{j\omega n})}_{\text{frequency response}} \quad -(3)$$

where $$x(n) = e^{j\omega n} \quad -(4)$$

Similarily, we can write

$$y(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{-j\omega n}) e^{j\omega n} \cdot H(e^{j\omega n}) \, d\omega \quad -(5)$$

We know that $$y(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} Y(e^{j\omega}) e^{j\omega n} \, d\omega \quad -(6)$$

Comparing the eq (5) & (6), we have

$$Y(e^{j\omega}) = X(e^{+j\omega n}) \cdot H(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})}$$

where $H(e^{j\omega})$ is known as transfer function of the system

→ Determine and sketch the magnitude and phase response of

$y(n) = \frac{1}{2}[x(n) + x(n-2)]$

Sol: Given $y(n) = \frac{1}{2}[x(n) + x(n-2)]$

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y(n) e^{-j\omega n}$$

$$= \frac{1}{2}\sum_{n=-\infty}^{\infty} [x(n) + x(n-2)] e^{-j\omega n}$$

$$= \frac{1}{2}\sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} + \frac{1}{2}\sum_{n=-\infty}^{\infty} x(n-2) e^{-j\omega n}$$

$$= \frac{1}{2}[X(e^{+j\omega}) + e^{-2j\omega} \cdot X(e^{+j\omega})]$$

$$Y(e^{j\omega}) = \frac{X(e^{-j\omega})}{2}[1 + e^{-2j\omega}]$$

$$\frac{Y(e^{j\omega})}{X(e^{+j\omega})} = H(e^{j\omega}) = \frac{1 + e^{-2j\omega}}{2} = \frac{1 + (\cos 2\omega - j\sin 2\omega)}{2}$$

$$H(e^{j\omega}) = \frac{1 + \cos 2\omega \mp j\sin 2\omega}{2}$$

We know $|H(e^{j\omega})|$ is symmetric and $\angle H(e^{j\omega})$ is anti-symmetric with a period of 2π in ω. We only need to know $H(e^{j\omega})$ over the interval $0 \le \omega \le 2\pi$

$$H(e^{j\omega}) = \frac{1}{2}\left[(1+\cos 2\omega)^2 + (\sin 2\omega)^2\right]^{1/2}$$

$$= \frac{1}{2}\left[1 + \cos^2 2\omega + \sin^2 2\omega + 2\cos 2\omega\right]^{1/2}$$

$$= \frac{1}{2}\left[1 + 1 + 2\cos 2\omega\right]^{1/2}$$

$$= \frac{1}{2}\left[2(1 + \cos 2\omega)\right]^{1/2}$$

$$= \frac{1}{2}\left[2 \cdot 2\,\cancel{\sin^2}\,\cos^2\omega\right]^{1/2}$$

$H(e^{j\omega}) = \cancel{\sin\omega}\ \cos\omega$

$$\angle H(e^{j\omega}) = \tan^{-1}\left(\frac{b}{a}\right)$$

$$= \tan^{-1}\left(\frac{-\sin 2\omega}{1+\cos 2\omega}\right)$$

$$= \tan^{-1}\left(\frac{-2\sin\omega\cos\omega}{2\cos^2\omega}\right)$$

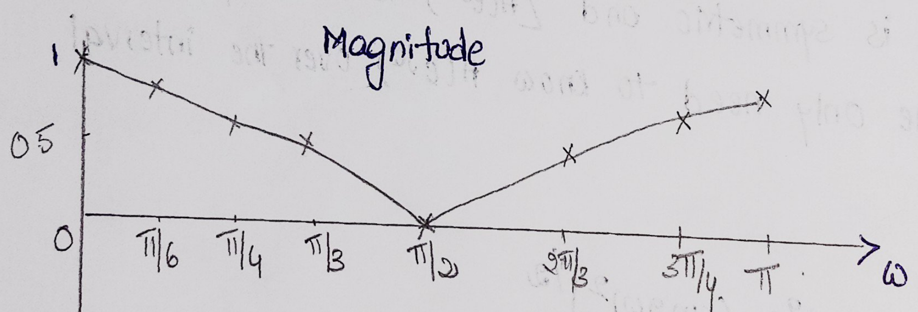
$$= \tan^{-1}(-\tan\omega)$$

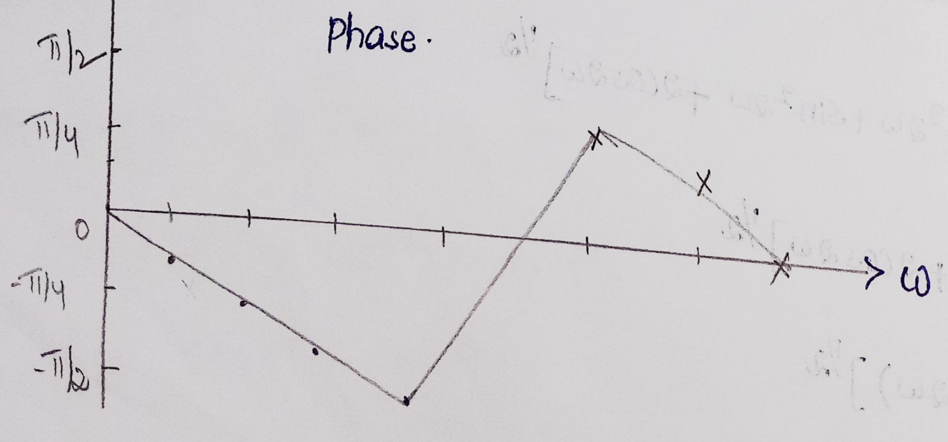
$$= -\omega$$

$\angle H(e^{j\omega}) = -\omega$ for $H(e^{j\omega}) > 0$

$= -\omega + \pi$ for $H(e^{j\omega}) < 0$

ω	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	π
$H(e^{j\omega})$	1	0.812	0.707	0.5	0	−0.5	−0.707	−1
$\lvert H(e^{j\omega})\rvert$	1	0.812	0.707	0.5	0	0.5	0.707	1
$\angle H(e^{j\omega})$	0	$-\pi/6$	$-\pi/4$	$-\pi/3$	$-\pi/2$	$\pi/3$	$\pi/4$	0





Previous paper question

→ A discrete-time system has a unit sample response h(n) given by $h(n) = \frac{1}{2}\delta(n) + \delta(n-1) + \frac{1}{2}\delta(n-2)$. Find the system frequency response $H(e^{j\omega})$. Plot magnitude and phase response

sol.

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n) e^{-j\omega n} \quad -①$$

$$= \sum_{n=-\infty}^{\infty} \left[\frac{1}{2}\delta(n) + \delta(n-1) + \frac{1}{2}\delta(n-2) \right] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{2}\delta(n) e^{-j\omega n} + \sum_{n=-\infty}^{\infty} \delta(n-1) e^{-j\omega n} + \frac{1}{2}\sum_{n=-\infty}^{\infty} \delta(n-2) e^{-j\omega n}$$

$$= \frac{1}{2} e^{-j\omega n}\Big|_{n=0} + e^{-j\omega n}\Big|_{n=1} + \frac{1}{2}\cdot e^{-j\omega n}\Big|_{n=2}$$

$$= \frac{1}{2} + e^{-j\omega} + \frac{1}{2} e^{-j2\omega} \quad -②$$

$$= e^{-j\omega}\left(\frac{1}{2} e^{j\omega} + 1 + \frac{1}{2} e^{-j\omega} \right)$$

$$= e^{-j\omega}\left(1 + \frac{1}{2}\left(e^{j\omega} + e^{-j\omega} \right) \right)$$

$$H(e^{j\omega}) = e^{-j\omega}(1 + \cos\omega) \quad -③$$

$$|H(e^{j\omega})| = |e^{-j\omega}||1+\cos\omega|$$

$$= 1+\cos\omega \quad (\because |e^{-j\omega}| = 1) \quad -④$$

From eq ②

$$H(e^{j\omega}) = \frac{1}{2} + e^{-j\omega} + \frac{1}{2} e^{-j2\omega}$$

$$= \frac{1}{2} + \cos\omega - j\sin\omega + \frac{1}{2}(\cos 2\omega - j\sin 2\omega)$$

$$= \frac{1}{2} + \cos\omega + \frac{1}{2}\cos 2\omega - j\sin\omega - \frac{j\sin 2\omega}{2}$$

$$= \cos\omega + \frac{1}{2}(1+\cos 2\omega) - j\left(\sin\omega + \frac{\sin 2\omega}{2}\right)$$

$$= \cos\omega + \frac{1}{\cancel{2}} \cdot \cancel{2}\cos^2\omega - j\left(\sin\omega + \frac{\cancel{2}\sin\omega\cos\omega}{\cancel{2}}\right)$$

$$= \cos\omega + \cos^2\omega - j\sin\omega(1+\cos\omega)$$

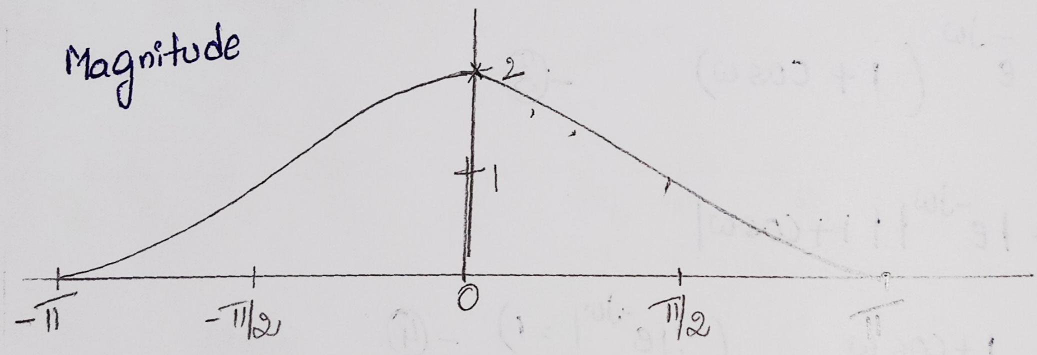
$$= \cos\omega(1+\cos\omega) - j\sin\omega(1+\cos\omega)$$

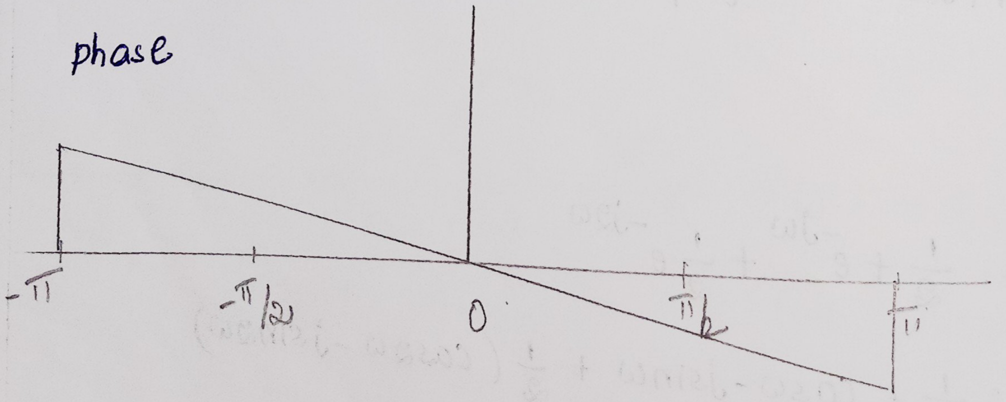
$$\angle H(e^{j\omega}) = \tan^{-1}\left(\frac{-\sin\omega(1+\cancel{\cos}\omega)}{\cos\omega(1+\cancel{\cos}\omega)}\right) \qquad H(e^{j\omega}) = 1+\cos\omega$$

$$= \tan^{-1}(-\tan\omega)$$

$$\angle H(e^{j\omega}) = -\omega \text{ for } 0 \le \omega \le \pi$$

ω	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	π
$H(e^{j\omega})$	2	1.866	1.707	1.5	1	0.5	0.292	0
$\lvert H(e^{j\omega})\rvert$	2	1.866	1.707	1.5	1	0.5	0.292	0
$\angle H(e^{j\omega})$	0	$-\pi/6$	$-\pi/4$	$-\pi/3$	$-\pi/2$	$-2\pi/3 \; \frac{\pi}{3}$	$\pi/4$	0





Z-TRANSFORMS

A Linear Time-Invarient discrete-time system is represented by difference equations. The direct solution of higher order difference equation is quite tedious and time consuming. so usually they are solved by In-Direct methods.

The z-transform has the advantage that it is a simple and systematic method and the complete solution can be obtained in one step and the initial conditions can be introduced in the beginning of the process itself.

The z-transform plays an important role in the analysis and representation of discrete-time Linear shift Invarient (LSI) systems.

The bilateral or two-sided z-transform of a discrete-time signal or a sequence $x(n)$ is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

where z is a complex variable

The one-sided or unilateral z-transform is defined as

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$$

If $x(n)=0$ for $n<0$, the one-sided and two sided z-transforms are equivalent.

In the z-domain, the convolution of two time domain signals is equivalent to multiplication of their corresponding z-transforms. This property simplifies the analysis of the response of an LTI system to various signals.

Region of convergence (ROC)

For any given sequence, the z-transform may or may not converge. The set of values of z or equivalently the set of points in z-plane for which X(z) converges is called the Region of convergence (ROC) of X(z).

In general ROC can be $R_{x-} < |z| < R_{x+}$ where R_{x-} can be as small as zero and R_{x+} can be as large as infinity.

If there is no value of z (i.e no point in the z-plane) for which X(z) converges, then the sequence x(n) is said to be havining no z-transform.

Advantages of z-transform

1. The z-transform converts the difference equations of a discrete-time system into Linear algebraic equations so that the analysis becomes easy and simple
2. Convolution in time-domain is converted into multiplication in z-Domain
3. z-transform exists for most of the signals for which Discrete-Time Fourier Transform (DTFT) does not exist.
4. Since the DTFT is nothing but the z-transform evaluated along the unit circle in the z-plane, the frequency response can be easily determined.

→ Relation between Discrete-time Fourier Transform (DTFT) and Z-Transform

The Discrete-Time Fourier Transform (DTFT) of a sequence x(n) is given by

$$X(e^{j\omega}) \text{ or } X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

For the existence of DTFT, the above summation should converge i.e $x(n)$ must be absolutely summable. The z-transform of the sequence $x(n)$ is given by

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

where z is complex variable and is given by $z = r.e^{j\omega}$

where r is radius of the circle

$$X(z) = X(r.e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)(re^{j\omega})^{-n}$$

$$= \sum_{n=-\infty}^{\infty} [x(n) r^{-n}] e^{-j\omega n}$$

For the existance of z-transform, the above summation should converge i.e $x(n)r^{-n}$ must be absolutely summable, i.e

$$\sum_{n=-\infty}^{\infty} |x(n) r^{-n}| < \infty$$

The above equation represents DTFT of a signal $x(n)r^{-n}$. Hence, we can say that the z-transform of $x(n)$ is same as the DTFT of $x(n)r^{-n}$ so for many sequences, the DTFT may not exist but the z-transform may exist. When $r=1$, the DTFT is same as the z-transform i.e the DTFT is nothing but the z-transform evaluated along the unit circle centred at the origin of the z-plane.

→ Prove that the sequences

(a) $x(n) = a^n u(n)$ and (b) $x(n) = -a^n u(-n-1)$

have the same $X(z)$ and differ only in ROC. Also plot their ROC's.

Sol. The given sequence $a^n u(n)$ is a causal infinite duration sequence, i.e

$$x(n) = \begin{cases} a^n, & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$\therefore Z[x(n)] = Z[a^n u(n)]$$

$$= \sum_{n=-\infty}^{\infty} a^n u(n) \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n \cdot z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

$$= 1 + az^{-1} + (az^{-1})^2 + \dots + \infty$$

$$= \frac{1}{1-az^{-1}}$$

$$= \frac{z}{z-a} \; ; \; ROC \; ; \; |z| > |a|$$

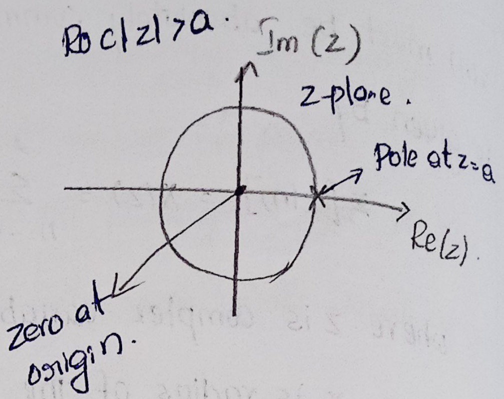


Fig1(a)

which implies that the ROC is exterior to the circle of radius 'a' as shown in fig1(a).

(b) The given signal $x(n) = -a^n u(-n-1)$ is a non-causal infinite duration sequence,

$$x(n) = \begin{cases} -a^n & n \le -1 \\ 0 & 0 \end{cases}$$

$$\therefore X(z) = \sum_{n=-\infty}^{\infty} -a^n u(-n-1) z^{-n} = \sum_{n=-\infty}^{-1} -a^n z^{-n} = \sum_{n=1}^{\infty} -a^{-n} z^n$$

$$= -\sum_{n=1}^{\infty} (a^{-1}z)^n$$

$$= -\left[\sum_{n=0}^{\infty} (a^{-1}z)^n - 1\right]$$

$$= 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n$$

$$= 1 - \left[1 + a^{-1}z + (a^{-1}z)^2 \dots \infty\right]$$

$$= 1 - \frac{1}{1-a^{-1}z} = \frac{1-a^{-1}z-1}{1-a^{-1}z}$$

$$= \frac{-a^{-1}z}{1-a^{-1}z} = \frac{z}{z-a} \; ; \; ROC \; ; \; |z| < |a|.$$

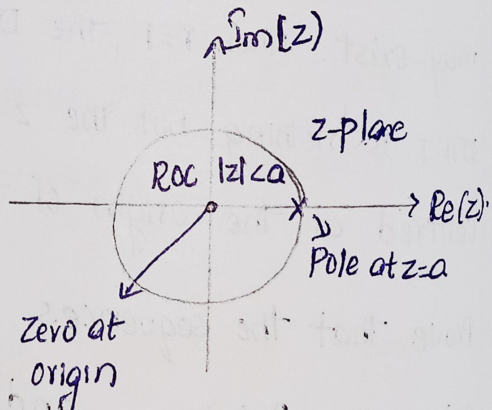


Fig 1(b)

That is, the ROC is the interior of the circle of radius 'a' as shown in Fig 1(b). From this example, we can observe that the z-transform of the sequences $a^n u(n)$ and $-a^n u(-n-1)$ are same, even though the sequences are different. Only ROC differentiates them. Therefore, to find the correct inverse z-transform, it is essential to know the ROC. The ROC are shown in figure 1(a) & 1(b).

→ Find the z-transform and ROC of the sequence $x(n) = \{2, 1, -3, 0, \underset{\uparrow}{4}, 3, 2, 1, 5\}$

Sol: The given sequence values are:

$x(-4) = 2$, $x(-3) = 1$, $x(-2) = -3$, $x(-1) = 0$, $x(0) = 4$, $x(1) = 3$, $x(2) = 2$, $x(3) = 1$, $x(4) = 5$

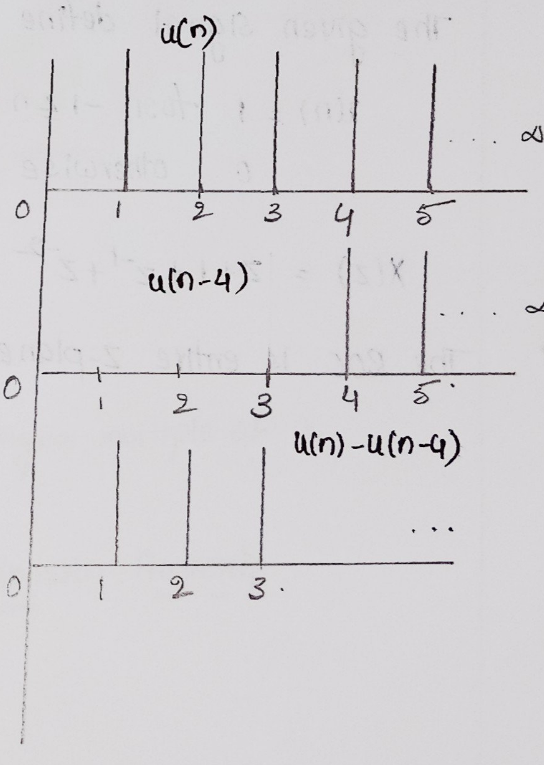
We know that $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$= 2z^4 + z^3 + -3z^2 + 4 + 3z^{-1} + 2z^{-2} + z^{-3} + 5z^{-4}$$

The ROC is entire z-plane except at $z = 0$ and $z = \infty$

→ Find the z-transform of the following sequences:

(a) $u(n) - u(n-4)$ (b) $u(-n) - u(-n-3)$ (c) $u(2-n) - u(-2-n)$



Sol: The given sequence is

$x(n) = u(n) - u(n-4)$

From the eq, the sequence exist at

$x(n) = 1$ for $0 \le n \le 3$

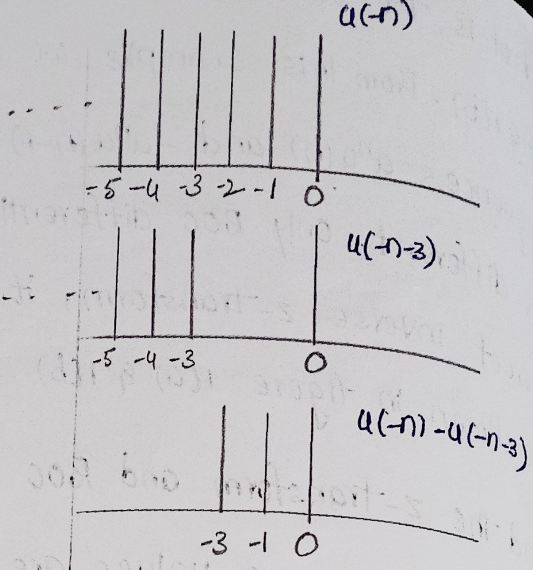
0

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= 1 + z^{-1} + z^{-2} + z^{-3}$$

The ROC is entire z-plane except at $z = 0$

b) $u(-n) - u(-n-3)$



Given sequence is

$x(n) = u(-n) - u(-n-3)$

$x(n) = 1$ for $-2 \le n \le 0$

$\quad\quad 0$ otherwise

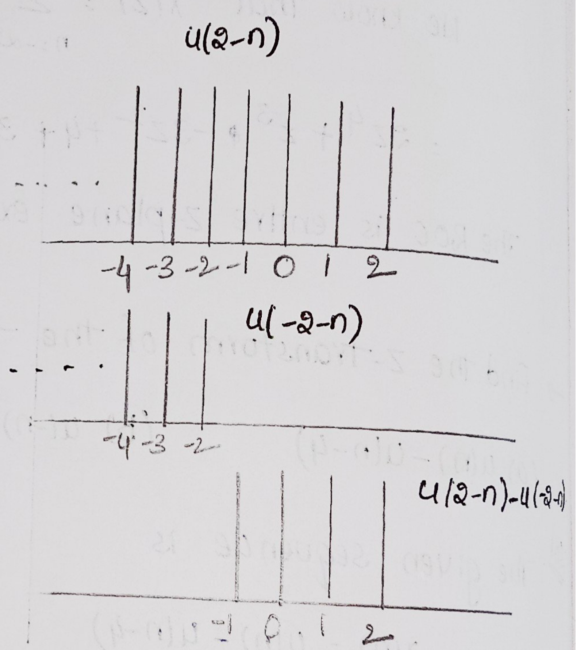
We know that $\sum_{n=-\infty}^{\infty} x(n) z^{-n} = X(z)$.

$$X(z) = \sum_{n=-2}^{0} x(n) z^{-n}$$

$$= z^2 + z + 1$$

The ROC is entire z-plane at $z = \infty$

c) $u(2-n) - u(-2-n)$



Given sequence is

$x(n) = u(2-n) - u(-2-n)$

We know that $\sum_{n=-\infty}^{\infty} x(n) z^{-n} = X(z)$.

The given signal define at

$x(n) = 1$ for $-1 \le n \le 2$

$\quad\quad 0$ otherwise

$$X(z) = z + 1 + z^{-1} + z^{-2}$$

The ROC is entire z-plane except at $z = 0$ & $z = \infty$

Properties of z-Transform

1. Linearity Property

If $x_1(n) \xleftrightarrow{Z.T} X_1(z)$, with ROC $= R_1$

and $x_2(n) \xleftrightarrow{Z.T} X_2(z)$, with ROC $= R_2$

Then $ax_1(n) + bx_2(n) \xleftrightarrow{Z.T} aX_1(z) + bX_2(z)$, with ROC$= R_1 \cap R_2$

2. Time Shifting Property

If $x_1(n) \xleftrightarrow{Z.T} X(z)$,

$x(n-m) \xleftrightarrow{Z.T} z^{-m} X(z)$

3. Multiplication by an Exponential Sequence property

If $x(n) \xleftrightarrow{Z.T} X(z)$

$a^n x(n) \xleftrightarrow{Z.T} X\left(\frac{z}{a}\right)$

4. Time Reversal Property

If $x(n) \xleftrightarrow{Z.T} X(z)$ with ROC$=R$

then $x(-n) \xleftrightarrow{Z.T} X\left(\frac{1}{z}\right)$ with ROC $= \frac{1}{R}$

5. Time Expansion Property

If $x(n) \xleftrightarrow{Z.T} X(z)$ with ROC $=R$

$x_k(n) \xleftrightarrow{Z.T} X(z^k)$, with ROC $= R^{1/k}$

where $x_k(n) = x\left(\frac{n}{k}\right)$ if n is an integer multiple of k

$= 0$, otherwise

6. Multiplication by 'n' or Differentiation in z-domain Property

If $x(n) \xleftrightarrow{Z.T} X(z)$, with ROC $= R$

$n.x(n) \xleftrightarrow{Z.T} -z\frac{d}{dz} X(z)$, with ROC $= R$

Common z-transform Pairs

	sequence	z-transform $X(z)$	ROC
1.	$\delta(n)$	1	All z
2.	$\delta(n-k)$	z^{-k}	All z except at $z=0$ (if $k>0$) All z except at $z=\infty$ (if $k<0$)
3.	$u(n)$	$\frac{z}{z-1}$	$\lvert z\rvert > 1$
4.	$a^n u(n)$	$\frac{z}{z-a}$	$\lvert z\rvert > \lvert a\rvert$
5.	$u(-n)$	$\frac{1}{1-z}$	$\lvert z\rvert < 1$
6.	$u(-n-1)$	$\frac{-z}{z-1}$	$\lvert z\rvert < 1$
7.	$u(-n-2)$	$\frac{-z^2}{z-1}$	$\lvert z\rvert < 1$
8.	$nu(n)$	$\frac{z}{(z-1)^2}$	$\lvert z\rvert > 1$
9.	$n.a^n u(n)$	$\frac{a.z}{(z-a)^2}$	$\lvert z\rvert > \lvert a\rvert$
10.	$\cos\omega n . u(n)$	$\frac{z(z-\cos\omega)}{z^2-2z\cos\omega+1}$	$\lvert z\rvert > 1$
11.	$a^n \cos\omega n \; u(n)$	$\frac{z(z-a\cos\omega)}{z^2-2az\cos\omega+a^2}$	$\lvert z\rvert > \lvert a\rvert$
12.	$(n+1)a^n u(n)$	$\frac{z^2}{(z-a)^2}$	$\lvert z\rvert > \lvert a\rvert$.

→ using properties of z-transform, find the z-transform of the following signals:

(a) $x(n) = u(-n)$ (c) $x(n) = u(-n-2)$

(b) $x(n) = u(-n+1)$ (d) $x(n) = 2^n u(n-2)$

(a) Given $x(n) = u(-n)$

We know that $z[u(n)] = \frac{z}{z-1} = \frac{1}{1-z^{-1}}$; ROC $|z|>1$

using the time-reversal property,

$$z[u(-n)] = \left.\frac{z}{z-1}\right|_{z=\frac{1}{z}} = \frac{\frac{1}{z}}{\frac{1}{z}-1} = \frac{1}{1-z} = \frac{-1}{1-z} \text{ ; ROC } |z|<1$$

(b). $x(n) = u(-n+1)$

$$X(z) = z[u(-n+1)] = z\{u[-(n-1)]\}$$

$$= z^{-1}.z[u(-n)]$$

$$= z^{-1}.\frac{1}{1-z} = \frac{1}{z(1-z)} = \frac{-1}{z(z-1)}$$

(c). Given $x(n) = u(-n-2) = u[-(n+2)]$

$$= z^2 z[u(-n)] = z^2.\frac{1}{1-z} = \frac{z^2}{1-z}$$

d) Given $x(n) = 2^n u(n-2)$

$$X(z) = z^{-2}.\frac{z}{z-1} = \frac{z^{-1}}{z-1} = \frac{1}{z(z-1)}$$

$$z[2^n u(n-2)] = z[u(n-2)]_{z=(z/2)}$$

$$= \left.\frac{1}{z(z-1)}\right|_{z=(z/2)} = \frac{1}{(\frac{z}{2})(\frac{z}{2}-1)} = \frac{4}{z(z-2)}$$

Inverse z-Transform

The process of finding the time domain signal $x(n)$ from its z-transform $X(z)$ is called the inverse z-transform which is denoted as

$$x(n) = z^{-1}[X(z)]$$

We have

$$X(z) = X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} [x(n)r^{-n}]e^{-j\omega n}$$

→ Find the inverse z-transform of

$$X(z) = z^3 + 2z^2 + z + 1 - 2z^{-1} - 3z^{-2} + 4z^{-3}$$

Sol. We know that

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = \dots + x(-3)z^3 + x(-2)z^2 + x(-1)z + x(0) + x(1)z^{-1} + x(2)z^{-2} + x(3)z^{-3}$$

Comparing this $X(z)$ with the given $X(z)$, we have

$$x(n) = \{1, 2, 1, \underset{\uparrow}{1}, -2, -3, 4\}$$

→ Find the inverse z-transform of

$$X(z) = \frac{z^{-1}}{3 - 4z^{-1} + z^{-2}} \; ; \; ROC; \; |z| > 1$$

Sol. Given $X(z) = \frac{z^{-1}}{3 - 4z^{-1} + z^{-2}} = \frac{z^{-1}}{z^{-2}(3z^2 - 4z + 1)} = \frac{z}{3z^2 - 4z + 1}$

$$= \frac{1}{3}\frac{z}{(z^2 - \frac{4}{3}z + 1)} =$$

$$\frac{X(z)}{z} = \frac{1}{3} \cdot \frac{1}{(z-1)(z-1/3)} = \frac{A}{z-1} + \frac{B}{z-1/3}$$

where A & B can be evaluated as follows :

$$A = (z-1)\cdot\frac{X(z)}{z}\Big|_{z=1} = (z-1)\cdot\frac{1}{3}\cdot\frac{1}{(z-1)(z-1/3)}\Big|_{z=1}$$

$$= \frac{1}{3}\cdot\frac{1}{(1-1/3)} = \frac{1}{3}\cdot\frac{3}{2} = \frac{1}{2}$$

$$B = \left(z-\frac{1}{3}\right)\cdot\frac{X(z)}{z}\Big|_{z=\frac{1}{3}} = \left(z-\frac{1}{3}\right)\cdot\frac{1}{3}\cdot\frac{1}{(z-1)(z-1/3)}\Big|_{z=\frac{1}{3}}$$

$$= \frac{1}{3}\cdot\frac{1}{\frac{1}{3}-1} = \frac{1}{3}\cdot\frac{3}{(-2)} = \frac{-1}{2}$$

$$\therefore \frac{X(z)}{z} = \frac{1}{2}\cdot\frac{1}{z-1} - \frac{1}{2}\cdot\frac{1}{(z-1/3)}$$

$$= \frac{1}{2}\left[\frac{1}{z-1} - \frac{1}{(z-1/3)}\right]$$

$$\therefore X(z) = \frac{1}{2}\left[\frac{z}{z-1} - \frac{z}{z-(1/3)}\right] \; ; \; ROC \; ; \; |z|>1$$

Since ROC is $|z|>1$, both the sequences are causal. Therefore taking inverse z-transform, we have

$$x(n) = \frac{1}{2}\left[u(n) - \left(\frac{1}{3}\right)^n u(n)\right] \; ; \; ROC \; ; \; |z|>1$$

→ **System Function**

Consider a discrete-time LTI system having an impulse response $h(n)$ as shown in Fig

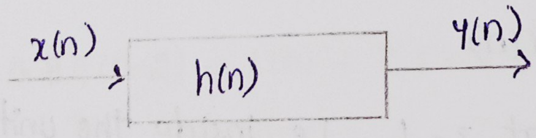


Fig: Discrete-time LTI system

Let us say it given an output $y(n)$ for an input $x(n)$, Then, we have

$$y(n) = x(n) * h(n)$$

Taking z-transform on both sides, we get

$$Y(z) = X(z)\,H(z)$$

where $Y(z)$ = z-transform of the output $y(n)$

$X(z)$ = z-transform of the input $x(n)$

$H(z)$ = z-transform of the impulse response $h(n)$

$$\therefore H(z) = \frac{Y(z)}{X(z)}$$

$H(z)$ is called the system function or the transfer function of the LTI discrete system and is defined as

The ratio of the z-transform of the output sequence $y(n)$ to the z-transform of the input sequence $x(n)$ when the initial conditions are neglected.

→ Consider an LTI system with a system function $H(z) = \frac{1}{1-(\frac{1}{2})z^{-1}}$. Find the difference equation. Determine the stability.

Sol

Given $H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-(\frac{1}{2})z^{-1}} = \frac{z}{z-\frac{1}{2}}$

$$Y(z) - \frac{1}{2}Y(z)\,z^{-1} = X(z)$$

Taking inverse z-transform on both sides we get the difference equation

$$y(n) - \frac{1}{2}y(n-1) = x(n)$$

The only pole of $H(z)$ is at $z = \frac{1}{2}$, i.e inside the unit circle. so the system is stable.

A causal system is represented by $H(z) = \frac{z+2}{2z^2-3z+4}$

Find the difference equation and the frequency response of the system.

Sol Given $H(z) = \frac{z+2}{2z^2-3z+4} = \frac{z(1+2z^{-1})}{z^2(2-3z^{-1}+4z^{-2})} = \frac{z^{-1}(1+2z^{-1})}{2-3z^{-1}+4z^{-2}}$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1}+2z^{-2}}{2-3z^{-1}+4z^{-2}}$$

$= 2Y(z) - 3z^{-1}Y(z) + 4z^{-2}Y(z) = z^{-1}X(z) + 2z^{-2}X(z)$

$= 2y(n) - 3y(n-1) + 4y(n-2) = x(n-1) + 2x(n-2)$

Which is the required difference equation

Putting $z = e^{j\omega}$ in $H(z)$, We get the frequency response $H(\omega)$ of the system

$$H(\omega) = \left.\frac{z+2}{2z^2-3z+4}\right|_{z=e^{j\omega}} = \frac{e^{j\omega}+2}{2(e^{j\omega})^2-3(e^{j\omega})+4} = \frac{e^{j\omega}+2}{2e^{j2\omega}-3e^{j\omega}+4}$$

$$= \frac{2 \cancel{e^{j\omega}} + \cos\omega + j\sin\omega}{2[\cos 2\omega + j\sin 2\omega] - 3[\sin\omega + j\sin\omega] + 4}$$

$$= \frac{(2+\cos\omega) + j\sin\omega}{2\cos 2\omega - 3\sin\omega + 4 + j(2\sin 2\omega - 3\sin\omega)}$$

→ Plot the pole-zero pattern and determine which of the following systems are stable :

(a) $y(n) = y(n-1) - 0.8y(n-2) + x(n) + x(n-2)$

(b) $y(n) = 2y(n-1) - 0.8y(n-2) + x(n) + 0.8x(n-1)$

Sol Given

$y(n) = y(n-1) - 0.8y(n-2) + x(n) + x(n-2)$

Taking z-transform on both sides and neglecting the initial conditions, We have

$$Y(z) = z^{-1}Y(z) - 0.8z^{-2}Y(z) + X(z) + z^{-2}X(z)$$

$$Y(z)(1 - z^{-1} + 0.8z^{-2}) = X(z)(1 + z^{-2})$$

$$\frac{Y(z)}{X(z)} = \frac{1+z^{-2}}{1-z^{-1}+0.8z^{-2}} = \frac{z^{-2}(z^2+1)}{z^{-2}(z^2-z+0.8)}$$

$$= \frac{z^2+1}{z^2-z+0.8} = \frac{(z+j)(z-j)}{(z-0.5-j0.74)(z-0.5+j0.74)}$$

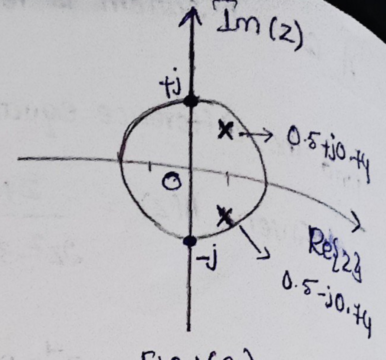


Fig. 1(a)

The zeros of H(z) are $z = \pm j1$ and $z = -j1$

The poles of H(z) are $z = 0.5 - j0.74$ and $z = 0.5 + j0.74$

The pole-zero plot is shown in Fig 1(a).

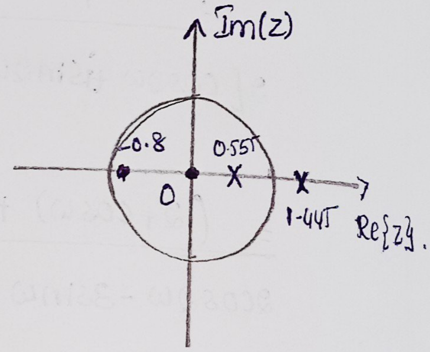
All the poles are inside the unit circle. Hence, the system is stable.

(b). Given $y(n) = 2y(n-1) - 0.8y(n-2) + x(n) + 0.8x(n-1)$

Taking z-transforms on both sides and neglecting initial conditions, we have

$$Y(z) = 2z^{-1}Y(z) - 0.8z^{-2}Y(z) + X(z) + z^{-1}0.8X(z)$$

$$Y(z)[1 - 2z^{-1} + 0.8z^{-2}] = X(z)(1 + 0.8z^{-1})$$



$$\frac{Y(z)}{X(z)} = \frac{1+0.8z^{-1}}{1-2z^{-1}+0.8z^{-2}}$$

$$= \frac{z^{-1}(z+0.8)}{z^{-2}(z^2-2z+0.8)} = \frac{z(z+0.8)}{z^2-2z+0.8}$$

$$= \frac{z(z+0.8)}{(z-1.445)(z-0.555)}$$

The zeros of H(z) are $z = 0$ and $z = -0.8$

The poles of H(z) are $z = 1.445$ and $z = 0.555$

The pole-zero plot is shown in fig 1(b)

One pole is outside the unit circle. Therefore the system is unstable

A Causal LTI system is described by the difference equation

$$y(n) = y(n-1) + y(n-2) + x(n) + 2x(n-1)$$

Find the system function and frequency response of the system. Plot the poles and zeros and indicate the ROC. Also determine the stability and impulse response of the system.

Sol. The given difference equation is

$$y(n) = y(n-1) + y(n-2) + x(n) + 2x(n-1)$$

Taking z-transform on both sides, we have

$$Y(z) = z^{-1}Y(z) + z^{-2}Y(z) + X(z) + 2z^{-1}X(z)$$

$$Y(z)\left[1 - z^{-1} - z^{-2}\right] = X(z)\left(1 + 2z^{-1}\right)$$

$$\frac{Y(z)}{X(z)} = \frac{1+2z^{-1}}{1-z^{-1}-z^{-2}} = \frac{z^{-1}(z+2)}{z^{-2}(z^2-z-1)} = \frac{z(z+2)}{z^2-z-1}$$

$$H(z) = \frac{z(z+2)}{z^2-z-1} = \frac{z(z+2)}{(z-1.62)(z+0.62)}$$

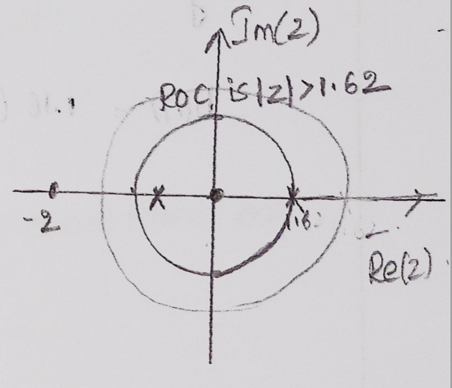
The frequeny response of the system is

$$H(\omega) = \frac{e^{j\omega}(e^{j\omega}+2)}{(e^{j\omega})^2 - e^{j\omega} - 1} = \frac{e^{j\omega}(e^{j\omega}+2)}{(e^{j\omega}-1.62)(e^{j\omega}+0.62)}$$

$H(z)$ has poles at $z = 1.62$ and $z = -0.62$. One of the pole is outside the unit circle. So the system is unstable.

$H(z)$ has zeros at $z=0$ and $z=-2$

The poles and zeros and the ROC as shown in fig.



To find the impulse response $h(n)$,

partial fraction expansion of $\frac{H(z)}{z}$ gives

$$\frac{H(z)}{z} = \frac{z+2}{(z-1.62)(z+0.62)} = \frac{A}{(z-1.62)} + \frac{B}{(z+0.62)}$$

$$A = (z-1.62) \cdot \frac{H(z)}{z} \Big|_{z=1.62}$$

$$= (z-\cancel{1.62}) \cdot \frac{(z+2)}{(z-\cancel{1.62})(z+0.62)} \Big|_{z=1.62}$$

$$= \frac{z+2}{z+0.62} = \frac{1.62+2}{1.62+0.62} = \frac{2.62}{2.24} = 1.16$$

$$B = (z+0.62) \cdot \frac{H(z)}{z} \Big|_{z=-0.62}$$

$$= (z+0.62) \cdot \frac{(z+2)}{(z-1.62)(z+0.62)} \Big|_{z=-0.62}$$

$$= \frac{z+2}{(z-1.62)} \Big|_{z=-0.62} = \frac{-0.62+2}{-0.62-1.62} = \frac{1.38}{-2.24} = -0.62$$

$$\therefore \frac{H(z)}{z} = \frac{1.16}{(z-1.62)} - \frac{0.62}{(z+0.62)}$$

$$H(z) = \frac{1.16z}{(z-1.62)} - 0.62 \frac{z}{(z+0.62)}$$

Taking inverse z-transforms, the impulse response is

$$h(n) = 1.16\,(1.62)^n u(n) - 0.62\,(-0.62)^n u(n)$$

Solution of Difference equations using Z-Transforms

To solve the difference equation, first it is converted into algebraic equation by taking its z-transform. The solution is obtained in z-domain and the time domain solution is obtained by taking its inverse z-Transform.

The system response has two components. The source free response and the forced response.

The response of the system due to input alone when the initial conditions are neglected is called the forced response of the system. It is also called the steady state response of the system.

The response of the system due to initial conditions alone when the input is neglected is called the free or natural response of the system. It is also called the transient response of the system.

The response due to input and initial conditions considered simultaneously is called the total response of the system.

→ A Linear shift invarient system is described by the difference equation

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n) + x(n-1) \quad \text{with}$$

$y(-1) = 0$ and $y(-2) = -1$. Find (a) the natural response of the system

(b) the forced response of the system for a step input

(c) The frequency response of the system.

Sol) The natural response is the response due to initial conditions only.

So $x(n) = 0$. Then the difference equation becomes

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = 0$$

Taking z-transform on both sides, we have

$$Y(z) - \frac{3}{4}z^{-1}Y(z) + \frac{1}{8}z^{-2}Y(z) = 0$$

$$Y(z) - \frac{3}{4}\left[z^{-1}Y(z) + y(-1)\right] + \frac{1}{8}\left[z^{-2}Y(z) + z^{-1}y(-1) + y(-2)\right] = 0$$

$$Y(z)-\frac{3}{4}\left[z^{-1}Y(z)+Y(-1)\right]+\frac{1}{8}\left[z^{-2}Y(z)+z^{-1}Y(-1)+Y(-2)\right]=0$$

$$Y(z)-\frac{3}{4}\left[z^{-1}Y(z)+0\right]+\frac{1}{8}\left[z^{-2}Y(z)+z^{-1}(0)-1\right]=0$$

$$Y(z)-\frac{3}{4}z^{-1}Y(z)+\frac{1}{8}z^{-2}Y(z)-\frac{1}{8}=0$$

$$Y(z)\left[1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}\right]-\frac{1}{8}=0$$

$$Y(z)=\frac{1/8}{1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2}}=\frac{\frac{1}{8}}{z^{-2}\left(z^{2}-\frac{3}{4}z+\frac{1}{8}\right)}$$

$$Y(z)=\frac{\frac{z^{2}}{8}}{\left(z^{2}-\frac{3}{4}z+\frac{1}{8}\right)}=\frac{z^{2}}{8\left(z^{2}-\frac{3}{4}z+\frac{1}{8}\right)}$$

$$\frac{Y(z)}{z}=\frac{1}{8}\cdot\frac{z}{\left(z^{2}-\frac{3}{4}z+\frac{1}{8}\right)}=\frac{1}{8}\left[\frac{z}{\left[z-(1/2)\right]\left[z-(1/4)\right]}\right]$$

Taking the partial fractions of above equation.

$$\frac{z}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}=\frac{A}{\left(z-\frac{1}{2}\right)}+\frac{B}{\left(z-\frac{1}{4}\right)}$$

$$A=\left(z-\frac{1}{2}\right)\cdot\left.\frac{Y(z)}{z}\right|_{z=1/2}$$

$$=\left.\cancel{\left(z-\frac{1}{2}\right)}\cdot\frac{z}{\cancel{\left(z-\frac{1}{2}\right)}\left(z-\frac{1}{4}\right)}\right|_{z=1/2}=\frac{\frac{1}{2}}{\frac{1}{2}-\frac{1}{4}}=\frac{\frac{1}{2}}{\frac{1}{4}}=2$$

$$B=\left(z-\frac{1}{4}\right)\cdot\left.\frac{Y(z)}{z}\right|_{z=1/4}$$

$$=\left.\cancel{\left(z-\frac{1}{4}\right)}\cdot\frac{z}{\left(z-\frac{1}{2}\right)\cancel{\left(z-\frac{1}{4}\right)}}\right|_{z=\frac{1}{4}}=\frac{\frac{1}{4}}{\frac{1}{4}-\frac{1}{2}}=\frac{\frac{1}{4}}{-\frac{1}{4}}=-1$$

$$\frac{Y(z)}{z} = \frac{1}{8}\left[\frac{2}{(z-1/2)} - \frac{1}{(z-1/4)}\right]$$

$$= \frac{1/4}{(z-1/2)} - \frac{1/8}{(z-1/4)}$$

$$Y(z) = \frac{1}{4}\cdot\frac{z}{(z-1/2)} - \frac{1}{8}\cdot\frac{z}{(z-1/4)}$$

$$y(n) = \frac{1}{4}\left(\frac{1}{2}\right)^n u(n) - \frac{1}{8}\left(\frac{1}{4}\right)^n u(n)$$

(b) To find the forced response due to a step input, put $x(n) = u(n)$. so, we have

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = u(n) + u(n-1)$$

We know that the forced response is due to input alone. so for forced response, the initial conditions are neglected. Taking z-transform on both sides of the above equation and neglecting the initial conditions, we have

$$Y(z) - \frac{3}{4}z^{-1}Y(z) + \frac{1}{8}z^{-2}Y(z) = \frac{z}{z-1} + \frac{1}{z-1}$$

$$Y(z)\left[1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right] = \frac{z+1}{z-1}$$

$$Y(z) = \frac{z+1}{z^{-2}(z-1)\left(z^2 - \frac{3}{4}z + \frac{1}{8}\right)} = \frac{z^2(z+1)}{(z-1)\left(z^2 - \frac{3}{4}z + \frac{1}{8}\right)}$$

$$\frac{Y(z)}{z} = \frac{z(z+1)}{(z-1)\left(z^2 - \frac{3}{4}z + \frac{1}{8}\right)} = \frac{z(z+1)}{(z-1)(z-0.5)(z-0.25)} = \frac{z(z+1)}{(z-1)(z-\frac{1}{2})(z-\frac{1}{4})}$$

Taking the partial fractions of $Y(z)/z$, we have

$$\frac{Y(z)}{z} = \frac{A}{(z-1)} + \frac{B}{(z-1/2)} + \frac{C}{(z-1/4)}$$

$$A = (z-1) \cdot \frac{z(z+1)}{(z-1)(z-1/2)(z-1/4)}\Bigg|_{z=1}$$

$$= \frac{1(2)}{(1-1/2)(1-1/4)} = \frac{2}{\frac{1}{2}\cdot\frac{3}{4}} = \frac{16}{3}$$

$$B = (z-1/2) \cdot \frac{z(z+1)}{(z-1)(z-1/2)(z-1/4)}\Bigg|_{z=1/2}$$

$$= \frac{\frac{1}{2}\left(\frac{1}{2}+1\right)}{\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-\frac{1}{4}\right)} = \frac{\frac{1}{2}\times\frac{3}{2}}{\frac{-1}{2}\left(\frac{1}{4}\right)} = -6$$

$$C = (z-1/4) \cdot \frac{z(z+1)}{(z-1)(z-1/2)(z-1/4)}\Bigg|_{z=1/4}$$

$$= \frac{\frac{1}{4}\left(\frac{1}{4}+1\right)}{\left(\frac{1}{4}-1\right)\left(\frac{1}{4}-\frac{1}{2}\right)} = \frac{\frac{1}{4}\times\frac{5}{4}}{\frac{-3}{4}\times\left(\frac{-1}{4}\right)} = \frac{5}{3}$$

$$\therefore \frac{Y(z)}{z} = \frac{16/3}{z-1} - \frac{6}{z-1/2} + \frac{5/3}{(z-1/4)}$$

$$Y(z) = \frac{16}{3}\cdot\frac{z}{z-1} - 6\,\frac{z}{z-1/2} + \frac{5}{3}\cdot\frac{z}{z-1/4}$$

$$= \frac{16}{3}u(n) - 6\left(\frac{1}{2}\right)^n u(n) + \frac{5}{3}\left(\frac{1}{4}\right)^n u(n)$$

(c) The frequency response of the system $H(\omega)$ is obtained by putting $z = e^{j\omega_{in}}$ $H(z)$

$$H(z) = \frac{Y(z)}{X(z)} =$$

The given difference equation is

$$y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n) + x(n-1)$$

Taking z-transforms on both sides, then

$$Y(z) - \frac{3}{4}z^{-1}Y(z) + \frac{1}{8}z^{-2}Y(z) = X(z) + z^{-1}X(z)$$

$$Y(z)\left[1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}\right] = X(z)\left[1+z^{-1}\right]$$

$$\frac{Y(z)}{X(z)} = \frac{(1+z^{-1})}{(1-\frac{3}{4}z^{-1}+\frac{1}{8}z^{-2})} = \frac{z^{-1}(z+1)}{z^{-2}(z^2-\frac{3}{4}z+\frac{1}{8})}$$

$$H(z) = \frac{z(z+1)}{z^2-\frac{3}{4}z+\frac{1}{8}}$$

putting $z = e^{j\omega}$ then we get frequency response, is

$$H(\omega) = \frac{e^{j\omega}(e^{j\omega}+1)}{(e^{j\omega})^2-\frac{3}{4}e^{j\omega}+\frac{1}{8}} = \frac{e^{j\omega}(e^{j\omega}+1)}{e^{2j\omega}-\frac{3}{4}e^{j\omega}+\frac{1}{8}}$$

Previous problem

2. Solve the following difference equation

$y(n) + 2y(n-1) = x(n)$ with $x(n) = (1/3)^n u(n)$ and the initial condition $y(-1) = 1$

Sol: The solution of the difference equation considering the initial condition and input simultaneously gives the total response of the system

The given difference equation is

$$y(n) + 2y(n-1) = x(n) = \left(\frac{1}{3}\right)^n u(n) \quad \text{with } y(-1) = 1$$

Taking z-transforms on both sides, we get

$$Y(z) + 2[z^{-1}Y(z) + y(-1)] = X(z) = \frac{z}{z-(1/3)}$$

$$Y(z)[1+2z^{-1}] + 2 = \frac{z}{z-1/3} = \frac{1}{1-(1/3)z^{-1}}$$

$$Y(z)(1+2z^{-1}) = -2 + \frac{1}{1-1/3z^{-1}}$$

$$Y(z) = \frac{-2}{(1+2z^{-1})} + \frac{1}{(1-\frac{1}{3}z^{-1})(1+2z^{-1})}$$

$$= \frac{-2z}{z+2} + \frac{z^2}{(z-1/3)(z+2)} \quad - \quad 1$$

Let $Y_1(z) = \dfrac{z^2}{(z-1/3)(z+2)}$

$$\frac{Y_1(z)}{z} = \frac{z}{(z-1/3)(z+2)} = \frac{A}{(z-1/3)} + \frac{B}{z+2}$$

$$A = (z-1/3)\cdot\frac{Y_1(z)}{z}\Big|_{z=1/3} = (z-1/3)\cdot\frac{z}{(z-1/3)(z+2)} = \frac{z}{z+2}\Big|_{z=1/3}$$

$$= \frac{\frac{1}{3}}{\frac{1}{3}+2} = \frac{\frac{1}{3}}{\frac{7}{3}} = \frac{1}{7}$$

$$B = (z+2)\cdot\frac{Y_1(z)}{z}\Big|_{z=-2} = \cancel{(z+2)}\cdot\frac{z}{(z-1/3)\cancel{(z+2)}}\Big|_{z=-2}$$

$$= \frac{-2}{-2-\frac{1}{3}} = \frac{-2}{-\frac{7}{3}} = \frac{6}{7}$$

$$\therefore \frac{Y_1(z)}{z} = \frac{z}{(z-1/3)(z+2)} = \frac{1}{7}\cdot\frac{1}{(z-1/3)} + \frac{6}{7}\cdot\frac{1}{(z+2)}$$

$$Y_1(z) = \frac{1}{7}\frac{z}{(z-1/3)} + \frac{6}{7}\cdot\frac{z}{(z+2)}$$

Substituting $Y_1(z)$ in eq ①, then we get

$$Y(z) = \frac{-2z}{z+2} + \frac{1}{7}\cdot\frac{z}{z-1/3} + \frac{6}{7}\cdot\frac{z}{(z+2)}$$

$$= \frac{z}{z+2}\left(-2+\frac{6}{7}\right) + \frac{1}{7}\cdot\frac{z}{z-1/3}$$

$$Y(z) = -\frac{8}{7}\cdot\frac{z}{z+2} + \frac{1}{7}\cdot\frac{z}{(z-1/3)}$$

Taking inverse z-transforms on both sides, the solution of difference equation is

$$y(n) = -\frac{8}{7}(-2)^n u(n) + \frac{1}{7}\cdot\left(\frac{1}{3}\right)^n u(n)$$